Apollonian Circle Packings, Integers, and Higher-Dimensional Sphere Packings

Edna Jones

Rutgers, The State University of New Jersey

Math/Stats Colloquium
Carleton College
September 22, 2020

About Me

From Fairfield, IA

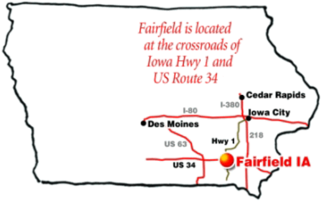


Figure: From http://www.jeffersoncountyiowa.com/.

Interested civil engineering

Figure: Clifton Suspension Bridge, Bristol, England.

About Me

ROSE-HULMAN INSTITUTE OF TECHNOLOGY

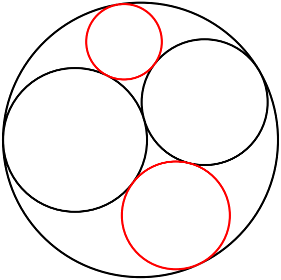
- Started as civil engineering major
- Switched to math by end of freshman year
- Interested in number theory
- Did a few internships & Research Experiences for Undergraduates (REUs)

About Me

- 5th year in mathematics Ph.D. program
- Like teaching mathematics
- Started getting involved in math outreach
  - Math Teachers' Circles
  - Julia Robinson Mathematics Festival (JRMF)
  - Math Circles

Apollonian Circle Packings: The Construction

Given three mutually tangent circles with disjoint points of tangency, there are exactly two circles tangent to the given ones. (Proved by Apollonius of Perga.)



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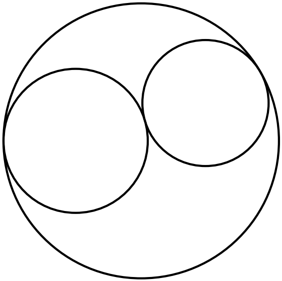

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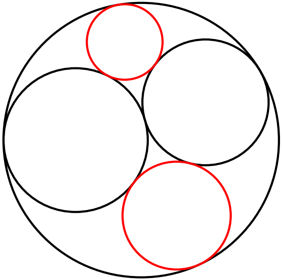

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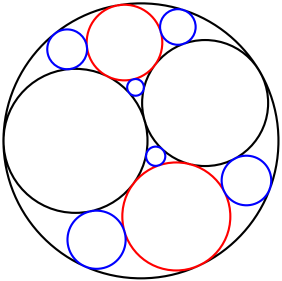

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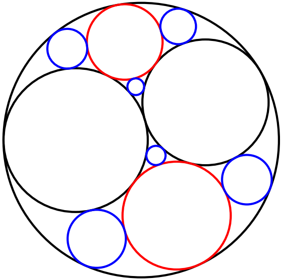


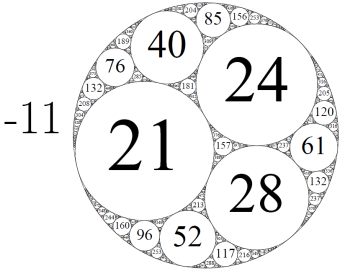
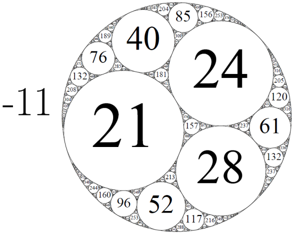


Figure: An Apollonian circle packing.

Apollonian Circle Packings



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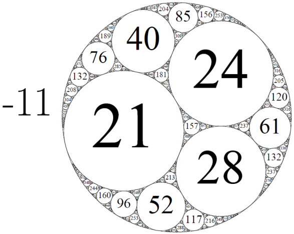
Label on circle:

bend = 1/radius



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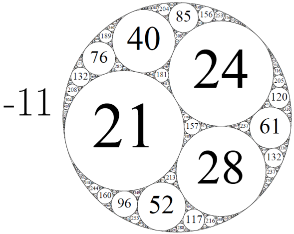

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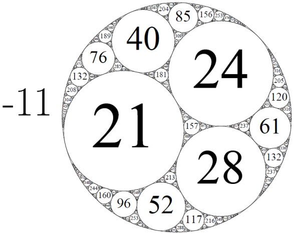

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Why?

“The Kiss Precise” by F. Soddy

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There’s now no need for rule of thumb.

Since zero bend’s a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Figure: An excerpt of “The Kiss Precise” by F. Soddy in *Nature*, 1936.

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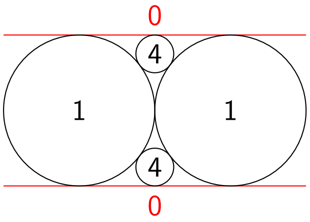

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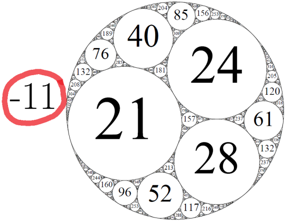

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If b_1, b_2, b_3, b_4 are bends of four mutually tangent circles, then

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2.$$

Descartes' Circle Theorem

Theorem (Descartes' Circle Theorem, 1643)

If b_1, b_2, b_3, b_4 *are bends of four mutually tangent circles, then*

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

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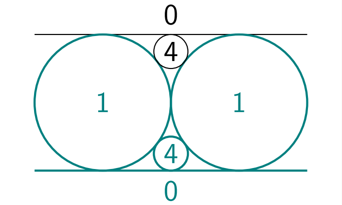

$b_1 = 0,\ b_2 = b_3 = 1,\ b_4 = 4$

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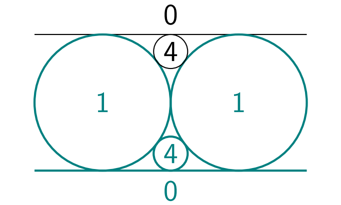

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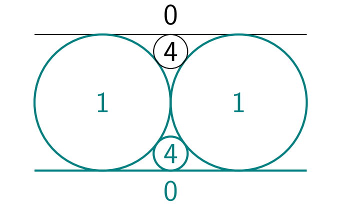

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$2(0^2 + 1^2 + 1^2 + 4^2) = 2(18) = 36$

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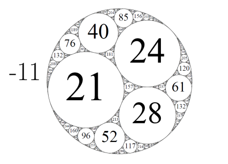

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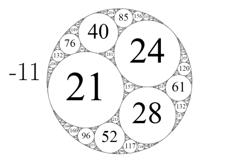

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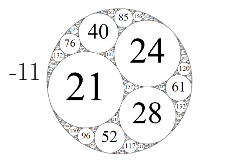

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Fix b_1, b_2, b_3. What do I know about the solutions to b_4?

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Fix b_1, b_2, b_3. What do I know about the solutions to b_4?

If b_4 and b_4' are solutions, b_1, b_2, b_3 fixed, then, by the quadratic formula,

$$b_4 + b_4' = 2(b_1 + b_2 + b_3).$$

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$

Matrix form:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}}_{M_4} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Matrices and Geometry

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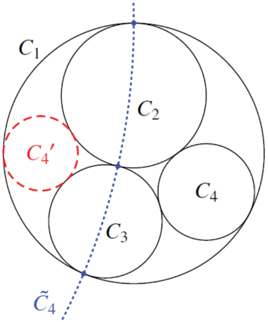

Figure: Four tangent circles and a reflection to a fifth circle.

$$M_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & & & \\ 2 & -1 & 2 & 2 \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 2 & 2 & -1 & 2 \\ & & & 1 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

Matrices and the Apollonian Group

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The *Apollonian group* $\Gamma := \langle M_1, M_2, M_3, M_4 \rangle$ (set of products of M_1, M_2, M_3, M_4)

- maps bends of an Apollonian circle packing to more bends of the packing,
- "generates" all bends of the packing from four bends, and
- sends integer vectors to integer vectors.

Integrality of Bends

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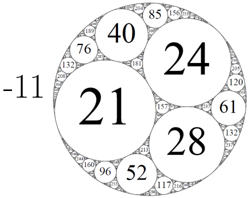

Figure: An Apollonian circle packing.

Since we start with an integer vector of bends
(namely, $(-11, 21, 24, 28)^t$),
all of our bends are integers!

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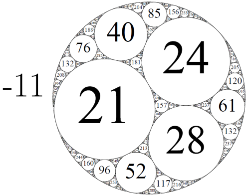


Figure: An Apollonian circle packing.

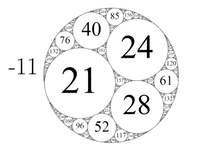
Which integers appear as bends?

Local Obstructions Modulo 24

Theorem (Fuchs, 2011)

For an integral, primitive Apollonian circle packing, there are local obstructions modulo 24 for the bends of the packing, that is, the remainder of a bend divided by 24 is in a certain subset of remainders.

Example



each bend

$$\equiv 0, 4, 12, 13, 16, \text{ or } 21 \pmod{24}.$$

Admissible Integers

Definition

An integer n is *admissible* if for every $q \geq 1$

$$n \equiv \text{bend of some circle in the packing} \pmod{q}$$

(that is, for every $q \geq 1$, there exists a bend m of a circle in the packing such that m and n have the same remainders when dividing by q).

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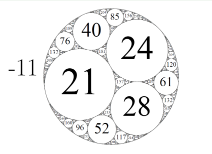


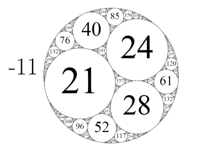
n is admissible $\iff$
$n \equiv 0, 4, 12, 13, 16,$ or $21 \pmod{24}$.

Local-Global Conjecture

Conjecture

The bends of a fixed primitive, integral Apollonian circle packing $\mathscr{P}$ satisfy a local-to-global principle.
That is, there is an $N_0 = N_0(\mathscr{P})$ so that, if $n > N_0$ and n is admissible, then n is the bend of a circle in the packing.

Example



We think that if
$n \equiv 0, 4, 12, 13, 16, \text{ or } 21 \pmod{24}$
and n is sufficiently large,
then n is the bend of a circle in the packing.
We do not have a proof of this!

Why Do We Have a Local-Global Conjecture?

The number of circles in $\mathscr{P}$ with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^δ, where $\delta =$ the Hausdorff dimension of the closure of $\mathscr{P}$.

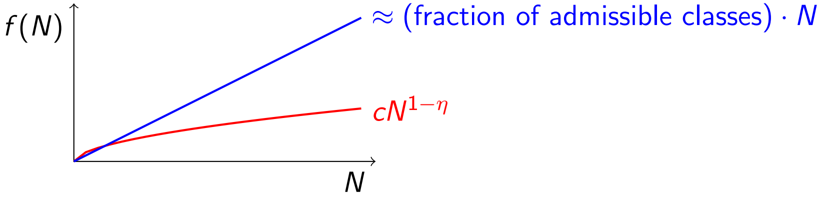
$$\delta \approx 1.30568\ldots$$

Thus, we would would expect that the multiplicity of a given admissible bend up to N is roughly $N^{\delta-1} \approx N^{0.30568} \geq 1$, so we should expect that every sufficiently large admissible number should be represented.

The Best We Can Do Right Now

Theorem (Bourgain–Kontorovich, 2014)

Almost every admissible number is the bend of a circle in the Apollonian circle packing $\mathscr{P}$. Quantitatively, the number of exceptions up to N is bounded above by $cN^{1-\eta}$, where $c, \eta > 0$ are constants only dependent on the packing.



Soddy Sphere Packings: The Construction

Given four mutually tangent spheres with disjoint points of tangency, there are exactly two spheres tangent to the given ones.

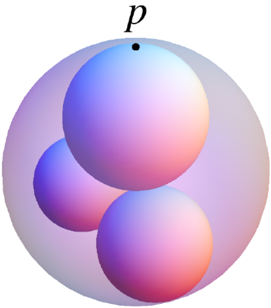


Figure: Four tangent spheres.

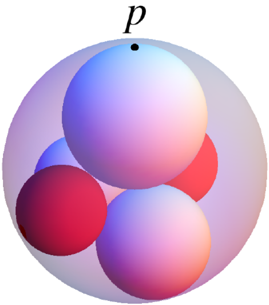
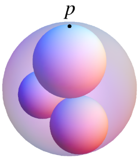


Figure: Four tangent spheres with two additional tangent spheres.



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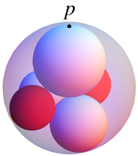

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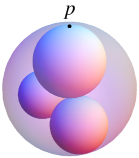

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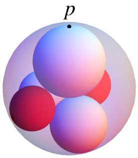

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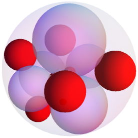

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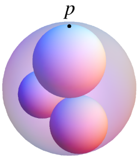

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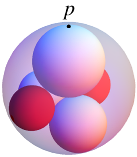

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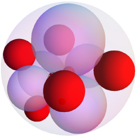

Figure: More tangent spheres.

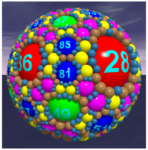

Figure: A Soddy sphere packing.

Soddy Sphere Packings

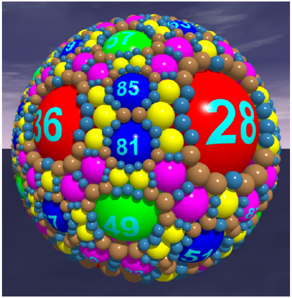

Figure: A Soddy sphere packing.

Label on sphere:
bend $= 1/\text{radius}$

What do you notice about the bends that you can see on this Soddy sphere packing?

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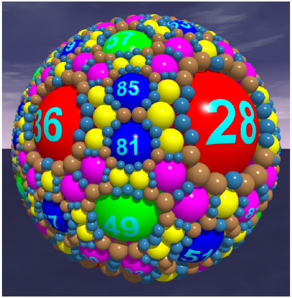

Figure: A Soddy sphere packing.

Label on sphere:

bend $= 1/\text{radius}$

What do you notice about the bends that you can see on this Soddy sphere packing?

They are all integers.

Why?

“The Kiss Precise” (Part 2)

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. Soddy.

Figure: The last stanza of “The Kiss Precise” by F. Soddy in *Nature*, 1936.

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If b_1, b_2, b_3, b_4, b_5 are bends of five mutually tangent spheres, then

$$(b_1 + b_2 + b_3 + b_4 + b_5)^2 = 3(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2).$$

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Fix b_1, b_2, b_3, b_4. What do I know about the solutions to b_5?

If b_5 and b_5' are solutions, b_1, b_2, b_3, b_4 fixed, then, by the quadratic formula,

$$b_5 + b_5' = b_1 + b_2 + b_3 + b_4.$$

$$b_5' = b_1 + b_2 + b_3 + b_4 - b_5$$

Matrix form:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}}_{M_5} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}$$

Matrices and Geometry

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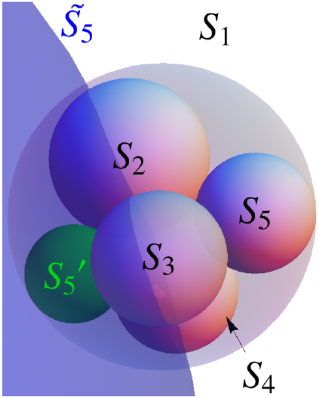

Figure: Five tangent spheres and a reflection to a sixth sphere

Matrices and the Soddy Group

$$M_1 = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & & & & \\ 1 & -1 & 1 & 1 & 1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ 1 & 1 & -1 & 1 & 1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$$

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The *Soddy group* $\Gamma := \langle M_1, M_2, M_3, M_4, M_5 \rangle$ (set of products of M_1, M_2, M_3, M_4, M_5)

- maps bends of a Soddy sphere packing to more bends of the packing,
- "generates" all bends of the packing from five bends, and
- sends integer vectors to integer vectors.

Integrality of Bends

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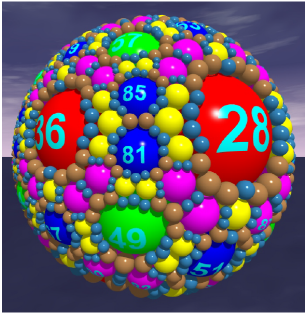
Since we start with an integer vector of five bends, all of our bends are integers!



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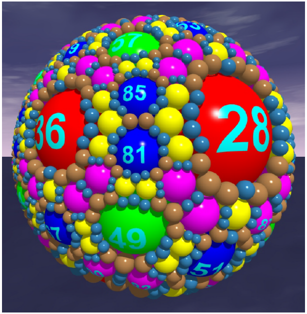


Figure: A Soddy sphere packing.

Which integers appear as bends?

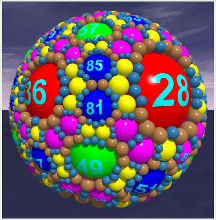
Local Obstructions Modulo 3

Lemma (Kontorovich, 2019)

For an integral, primitive Soddy sphere packing $\mathscr{P}$, there is an $\varepsilon = \varepsilon(\mathscr{P}) \in \{1, 2\}$ such that each bend of the packing is

$$\equiv 0 \text{ or } \varepsilon \pmod 3.$$

Example



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Admissible Integers

Definition

An integer n is *admissible* if for every $q \geq 1$

$$n \equiv \text{bend of some sphere in the packing} \pmod{q}$$

(that is, for every $q \geq 1$, there exists a bend m of a sphere in the packing such that m and n have the same remainders when dividing by q).

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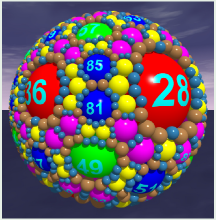


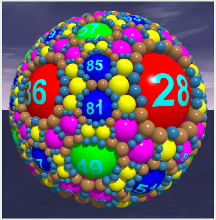
n is admissible $\iff$
$n \equiv 0$ or $1 \pmod{3}$.

Local-Global Theorem

Theorem (Kontorovich, 2019)

The bends of a fixed primitive, integral Soddy sphere packing $\mathscr{P}$ satisfy a local-to-global principle.
That is, there is an $N_0 = N_0(\mathscr{P})$ so that, if $n > N_0$ and n is admissible, then n is the bend of a sphere in the packing.

Example



If $n \equiv 0$ or $1 \pmod 3$ and n is sufficiently large, then n is the bend of a sphere in the packing.

Why Should We Have a Local-Global Theorem?

The number of spheres in $\mathscr{P}$ with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^{δ}, where $\delta =$ the Hausdorff dimension of the closure of $\mathscr{P}$.

$$\delta \approx 2.4739\ldots$$

Thus, we would would expect that the multiplicity of a given admissible bend up to N is roughly $N^{\delta-1} \approx N^{1.4739} \geq 1$, so we should expect that every sufficiently large admissible number should be represented.

Prove a local-global principle for bends of more general sphere packings (called crystallographic sphere packings).

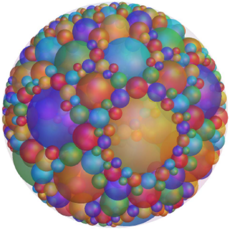


Figure: A crystallographic (more specifically, an orthoplicial) packing made by Kei Nakamura. Proven by Dias and Nakamura to have a local-global principle for bends.

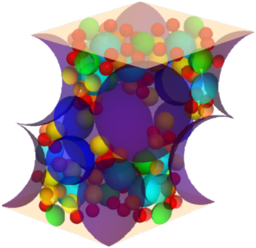


Figure: A fundamental domain of a crystallographic packing made by Arseniy (Senia) Sheydvasser. Not yet proven to have a local-global principle for bends.

Main References

- Jean Bourgain and Alex Kontorovich, "On the local-global conjecture for integral Apollonian gaskets," *Inventiones mathematicae*, volume 196, pp. 589–650, 2014.

  Some pictures used in this presentation related to Apollonian circle packings are from this paper.
- Alex Kontorovich, "From Apollonius to Zaremba: Local-global phenomena in thin orbits," *Bulletin of the American Mathematical Society*, volume 50, number 2, pp. 187-228, 2013, `https://www.ams.org/journals/bull/2013-50-02/S0273-0979-2013-01402-2/`.

  Some pictures used in this presentation related to Apollonian circle packings are from this paper.
- Alex Kontorovich, "The Local-Global Principle for Integral Soddy Sphere Packings," *Journal of Modern Dynamics*, volume 15, pp. 209-236, 2019, `https://www.aimsciences.org/article/doi/10.3934/jmd.2019019`.

  Most pictures used in this presentation related to Soddy sphere packings are from this paper.

Thank you for listening!

Proof Outline for Bourgain's and Kontorovich's Apollonian Circle Packing Result

1. Show that Γ contains the congruence subgroup $\Gamma(2)$ of $\mathrm{PSL}_2(\mathbb{Z})$. This shows that the set of bends contains the primitive values of a shifted binary (2-variable) quadratic form.
2. This shifted binary quadratic form gives you enough to work with so that you can apply the circle method (with some other tools used in the major arc and minor arc analyses) to obtain an "almost all" statement.

Proof Outline for Kontorovich's Soddy Sphere Packing Result

1. Show that Γ contains a congruence subgroup Ξ of $SL_2(\mathbb{C})$. This shows that the set of bends contains the "primitive" values of a shifted quaternary (4-variable) quadratic form.
2. This shifted quaternary quadratic form gives you enough to work with so that you can apply the circle method to show that every sufficiently large admissible number is represented as a bend.

Lemma

Γ contains (up to an isomorphism) the congruence subgroup

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathsf{PSL}_2(\mathcal{O}) : \beta, \gamma \equiv 0 \pmod{\varrho} \right\},$$

where $\mathcal{O} = \mathbb{Z}[\omega]$, $\omega = e^{\pi i/3}$, and $\varrho = 1 + \omega$.