Apollonian circle packings, integers, and higher-dimensional sphere packings

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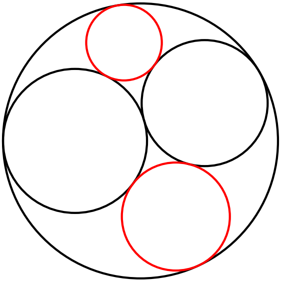
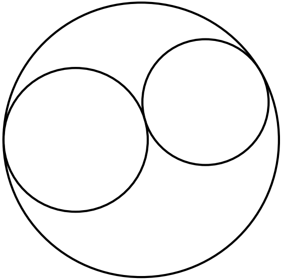
Colloquium
Spelman College
February 24, 2025

Given three pairwise tangent circles with disjoint points of tangency, there are exactly two circles tangent to the given ones. (Proved by Apollonius of Perga.)



Apollonian circle packings: The construction



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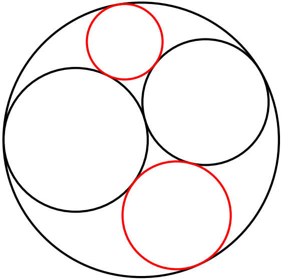

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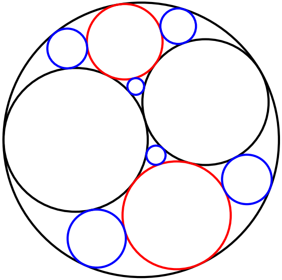

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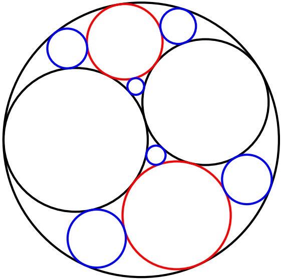


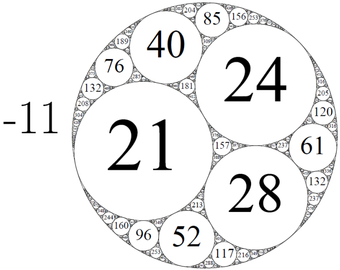


Figure: An Apollonian circle packing.

Apollonian circle packings

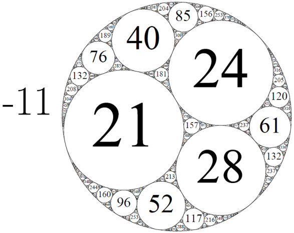
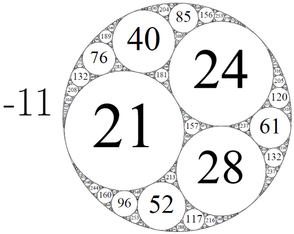


Figure: An Apollonian circle packing.

Label on circle:
bend = 1/radius

Apollonian circle packings



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Label on circle:

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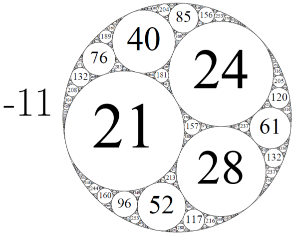
What do you notice about the bends that you can see in this Apollonian circle packing?

Apollonian circle packings



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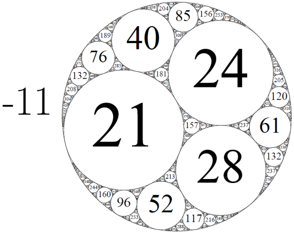

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What do you notice about the bends that you can see in this Apollonian circle packing?

They are all integers.

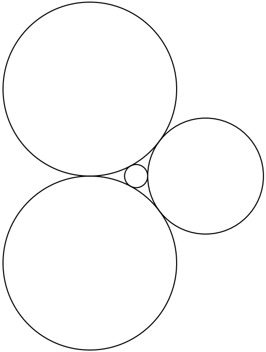
Why?

"The Kiss Precise" by F. Soddy

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.

Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Figure: An excerpt of "The Kiss Precise" by F. Soddy in *Nature*, 1936.



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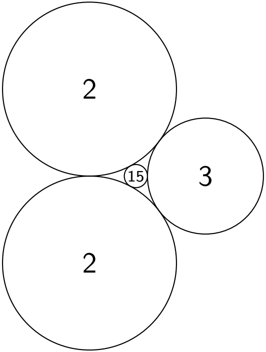

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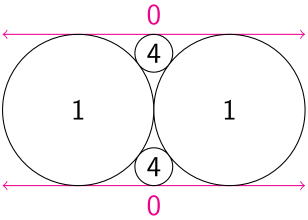

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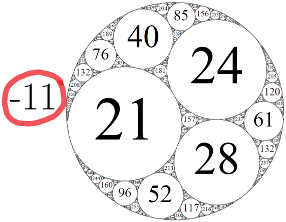

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If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2.$$

Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 *are bends of four pairwise tangent circles, then*

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

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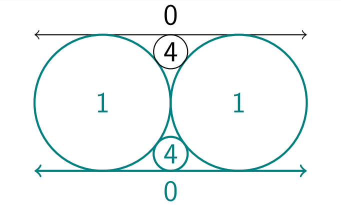

$b_1 = 0,\ b_2 = b_3 = 1,\ b_4 = 4$

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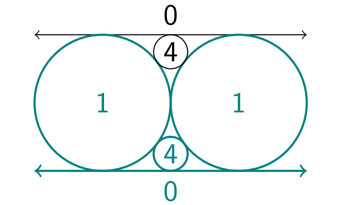

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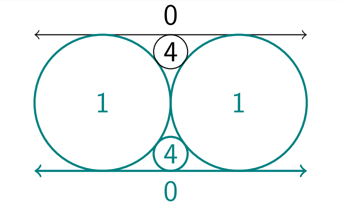

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$2(0^2 + 1^2 + 1^2 + 4^2) = 2(18) = 36$

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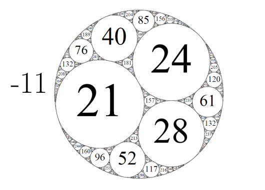

$$b_1 = -11,\ b_2 = 21,\ b_3 = 24,\ b_4 = 28$$

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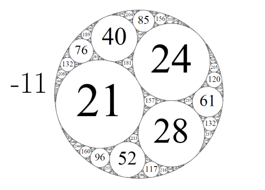

$$b_1 = -11,\ b_2 = 21,\ b_3 = 24,\ b_4 = 28$$

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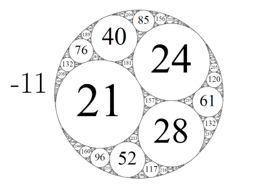

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Fix b_1, b_2, b_3. What do we know about the solutions for b_4?

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Fix b_1, b_2, b_3. What do we know about the solutions for b_4?

If b_4 and b_4' are solutions for fixed b_1, b_2, b_3, then, by the quadratic formula,

$$b_4 + b_4' = 2(b_1 + b_2 + b_3).$$

Matrices and geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$

Matrix form:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}}_{M_4} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

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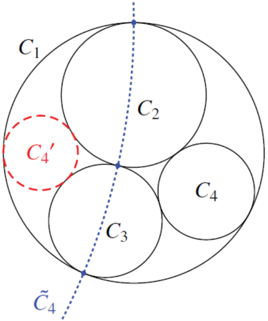

Figure: Four tangent circles and a reflection to a fifth circle.

Matrices and the Apollonian group

$$M_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & & & \\ 2 & -1 & 2 & 2 \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 2 & 2 & -1 & 2 \\ & & & 1 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

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The **Apollonian group** $\Gamma := \langle M_1, M_2, M_3, M_4 \rangle$ (set of products of M_1, M_2, M_3, M_4)

- maps bends in an Apollonian circle packing to more bends in the packing,
- "generates" all bends in the packing from four bends, and
- sends integer vectors to integer vectors.

Integrality of bends

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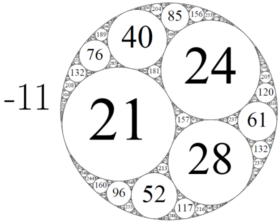
Since we started with an integer vector of bends (namely, $(-11, 21, 24, 28)^\top$), all of our bends are integers!



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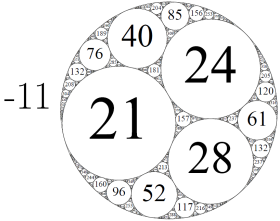

Which integers appear as bends?

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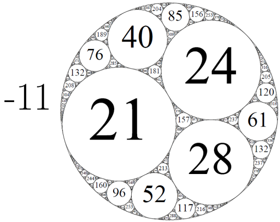


Which integers appear as bends?

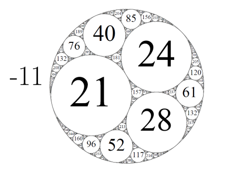
Are there any congruence or local obstructions?

Local obstructions modulo 24

Theorem (Fuchs, 2011)

For an integral, primitive Apollonian circle packing, there are local obstructions modulo 24 for the bends in the packing.
That is, the remainder of a bend divided by 24 is in a certain subset of remainders.
(The local obstructions depend on the packing.)

Example



each bend
$\equiv 0, 4, 12, 13, 16, \text{or } 21 \pmod{24}.$

Admissible integers

Definition (Admissible integers for Apollonian circle packings)

Let $\mathcal{P}$ be an integral Apollonian circle packing.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some circle in } \mathcal{P} \pmod{q}.$$

(That is, for every $q \geq 1$, there exists a circle in the packing with bend n such that m and n have the same remainders when dividing by q.)

Equivalently, m is admissible if m has no local obstructions to being the bend of a circle in the packing.

Admissible integers

Theorem (Fuchs, 2011)

An integer m is admissible if and only if m is in certain congruence classes modulo 24 (i.e., the remainder of m divided by 24 is in a certain subset of remainders).
(The congruence classes depend on the packing.)

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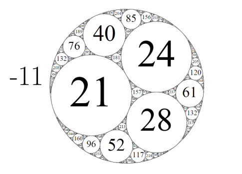


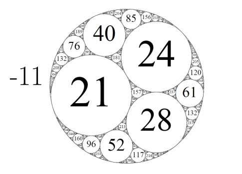
m is admissible $\iff$
$m \equiv 0, 4, 12, 13, 16, \text{or } 21 \pmod{24}$.

(Strong asymptotic) local-global conjecture

Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, 2003)

The bends in a fixed primitive, integral Apollonian circle packing $\mathcal{P}$ satisfy a (strong asymptotic) local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a circle in the packing.

Example



Mathematicians thought that if $m \equiv 0, 4, 12, 13, 16, \text{or } 21 \pmod{24}$ and m is sufficiently large, then m is the bend of a circle in the packing.

We do not have a proof of this!

Why did we have a (strong asymptotic) local-global conjecture?

Theorem (Kontorovich–Oh, 2011)

The number of circles in an Apollonian circle packing $\mathcal{P}$ with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^{δ}, where $\delta =$ the Hausdorff dimension of the closure of $\mathcal{P}$.

For Apollonian circle packings, we have

$$\delta \approx 1.30568\ldots$$

Thus, we would expect the multiplicity of a given admissible bend up to N to be roughly $N^{\delta-1} \approx N^{0.30568} \geq 1$, so we expected every sufficiently large admissible number to be represented.

(Strong asymptotic) local-global conjecture

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Multiple people worked towards proving this conjecture, including Bourgain, Fuchs, and Kontorovich.

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Multiple people worked towards proving this conjecture, including Bourgain, Fuchs, and Kontorovich.

Then this paper came out:

THE LOCAL-GLOBAL CONJECTURE FOR APOLLONIAN CIRCLE PACKINGS IS FALSE

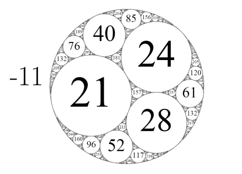
SUMMER HAAG, CLYDE KERTZER, JAMES RICKARDS, AND KATHERINE E. STANGE

The (strong asymptotic) local-global conjecture is false

Theorem (Haag–Kertzer–Rickards–Stange, 2024)

Certain quadratic and quartic families of bends (of the form cn^2 and cn^4 for a fixed integer c) are missing from some Apollonian circle packings.

Example



An integer of the form n^2 or $3n^2$ (where n is an integer) cannot appear as the bend of a circle in this packing.

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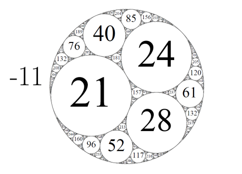


An integer of the form n^2 or $3n^2$ (where n is an integer) cannot appear as the bend of a circle in this packing.

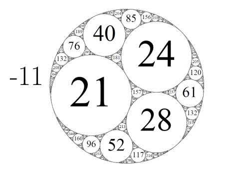
There are infinitely many admissible integers $m \equiv 0, 4, 12, 16 \pmod{24}$ that are of the form n^2 or $3n^2$.

Updated (strong asymptotic) local-global conjecture

Conjecture (Haag–Kertzer–Rickards–Stange, 2024)

There is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$, m is admissible and is not obstructed by any known quadratic or quartic obstructions, then m is the bend of a circle in the packing.

Example



It is now thought that if $m \equiv 0, 4, 12, 13, 16, \text{or } 21 \pmod{24}$, m is sufficiently large, and m is not of the form n^2 or $3n^2$, then m is the bend of a circle in the packing.

We do not have a proof of this!

Soddy sphere packings: The construction

Given four pairwise tangent spheres with disjoint points of tangency, there are exactly two spheres tangent to the given ones.

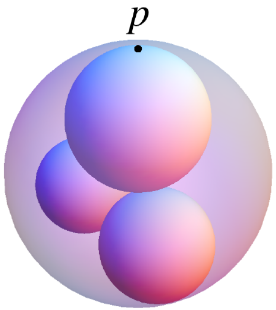


Figure: Four tangent spheres.

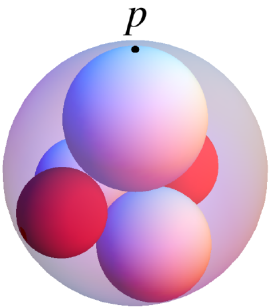
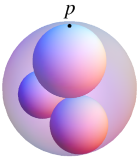


Figure: Four tangent spheres with two additional tangent spheres.

Soddy sphere packings: The construction



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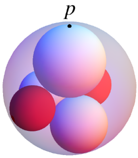

Figure: Four tangent spheres with two more spheres.

Soddy sphere packings: The construction

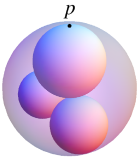

Figure: Four tangent spheres.

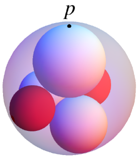

Figure: Four tangent spheres with two more spheres.

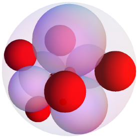

Figure: More spheres.

Soddy sphere packings: The construction

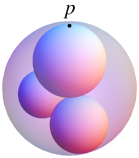

Figure: Four tangent spheres.

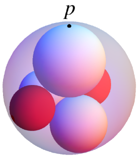

Figure: Four tangent spheres with two more spheres.

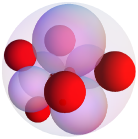

Figure: More spheres.

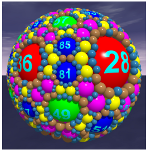

Figure: A Soddy sphere packing.

Soddy sphere packings

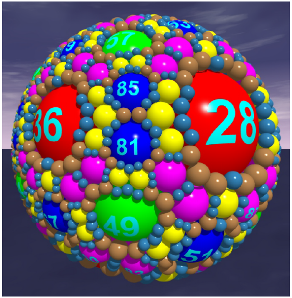

Figure: A Soddy sphere packing.

Label on sphere:

bend = 1/radius

What do you notice about the bends that you can see in this Soddy sphere packing?

Soddy sphere packings

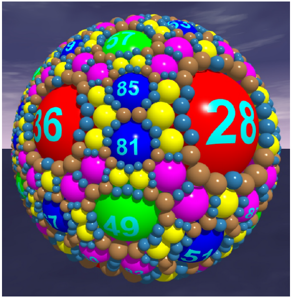

Figure: A Soddy sphere packing.

Label on sphere:
bend = 1/radius

What do you notice about the bends that you can see in this Soddy sphere packing?

They are all integers.

Why?

“The Kiss Precise” (Part 2)

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. Soddy.

Figure: The last stanza of “The Kiss Precise” by F. Soddy in *Nature*, 1936.

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Figure: The last stanza of "The Kiss Precise" by F. Soddy in *Nature*, 1936.

If b_1, b_2, b_3, b_4, b_5 are bends of five pairwise tangent spheres, then

$$(b_1 + b_2 + b_3 + b_4 + b_5)^2 = 3(b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2).$$

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Fix b_1, b_2, b_3, b_4. What do we know about the solutions for b_5?

If b_5 and b_5' are solutions for fixed b_1, b_2, b_3, b_4, then, by the quadratic formula,

$$b_5 + b_5' = b_1 + b_2 + b_3 + b_4.$$

Matrices and geometry

$$b_5' = b_1 + b_2 + b_3 + b_4 - b_5$$

Matrix form:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}}_{M_5} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}$$

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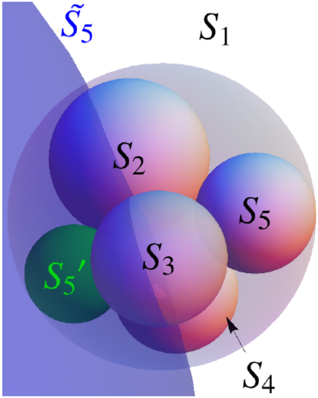

Figure: Five tangent spheres and a reflection to a sixth sphere.

Matrices and the Soddy group

$$M_1 = \begin{pmatrix} -1 & 1 & 1 & 1 & 1 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & & & & \\ 1 & -1 & 1 & 1 & 1 \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ 1 & 1 & -1 & 1 & 1 \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ 1 & 1 & 1 & -1 & 1 \\ & & & & 1 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

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The **Soddy group** $\Gamma := \langle M_1, M_2, M_3, M_4, M_5 \rangle$ (set of products of M_1, M_2, M_3, M_4, M_5)

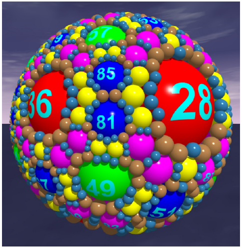
- maps bends in a Soddy sphere packing to more bends in the packing,
- "generates" all bends in the packing from five bends, and
- sends integer vectors to integer vectors.

Integrality of bends

The **Soddy group** $\Gamma := \langle M_1, M_2, M_3, M_4, M_5 \rangle$

- maps bends in a Soddy sphere packing to more bends in the packing,
- "generates" all bends in the packing from five bends, and
- sends integer vectors to integer vectors.

Since we started with an integer vector of five bends (namely, $(-11, 21, 25, 27, 28)^\top$), all of our bends are integers!



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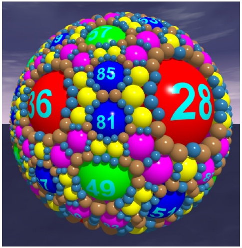

Which integers appear as bends?

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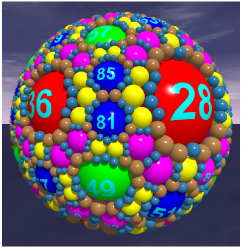


Which integers appear as bends?

Are there any congruence or local obstructions?

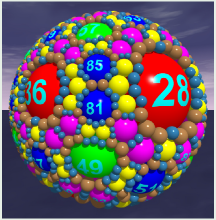
Local obstructions modulo 3

Lemma (Kontorovich, 2019)

For an integral, primitive Soddy sphere packing $\mathcal{P}$, there is an $\varepsilon(\mathcal{P}) \in \{1, 2\}$ such that each bend of the packing is

$$\equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod 3 .$$

Example



each bend $\equiv 0$ or $1 \pmod 3$.

Admissible integers

Definition (Admissible integers for Soddy sphere packings)

Let $\mathcal{P}$ be an integral Soddy sphere packing.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some sphere in } \mathcal{P} \pmod{q}.$$

(That is, for every $q \geq 1$, there exists a sphere in the packing with bend n such that m and n have the same remainders when dividing by q.)

Equivalently, m is admissible if m has no local obstructions to being the bend of a sphere in the packing.

Admissible integers

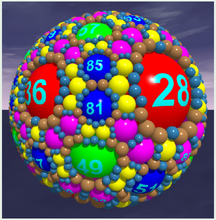
Theorem (Kontorovich, 2019)

m is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod 3,$$

where $\varepsilon(\mathcal{P}) \in \{1, 2\}$ depends only on the packing.

Example



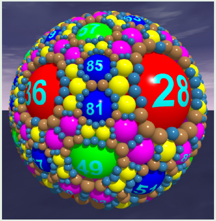
m is admissible $\iff$
$m \equiv 0$ or $1 \pmod 3$.

A (strong asymptotic) local-global principle

Theorem (Kontorovich, 2019)

The bends in a fixed primitive integral Soddy sphere packing $\mathcal{P}$ satisfy a (strong asymptotic) local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a sphere in the packing.

Example



If $m \equiv 0$ or $1 \pmod 3$ and m is sufficiently large, then m is the bend of a sphere in the packing.

Why should we have a (strong asymptotic) local-global principle?

Theorem (Kim, 2015)

Let $\mathcal{P}$ be a certain type of $(n-1)$-sphere packing (called a Kleinian sphere packing) in dimension $n \geq 2$.
The number of spheres in $\mathcal{P}$ with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^{δ}, where $\delta =$ the Hausdorff dimension of the closure of $\mathcal{P}$.

For Soddy sphere packings, we have

$$\delta \approx 2.4739\ldots$$

Thus, we would expect the multiplicity of a given admissible bend up to N to be roughly $N^{\delta-1} \approx N^{1.4739} \geq 1$, so we should expect every sufficiently large admissible number to be represented.

(Strong asymptotic) local-global principles

Goal: Prove (strong asymptotic) local-global principles for certain integral Kleinian sphere packings, that is, prove:
If m is admissible and sufficiently large, then m is the bend of an $(n-1)$-sphere in the packing.

Definition (Admissible integers)

Let $\mathcal{P}$ be an integral Kleinian sphere packing.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some } (n-1)\text{-sphere in } \mathcal{P} \pmod{q}.$$

(That is, for every $q \geq 1$, there exists an $(n-1)$-sphere in the packing with bend n such that m and n have the same remainders when dividing by q.)

I am working on (strong asymptotic) local-global principles for certain integral Kleinian sphere packings.

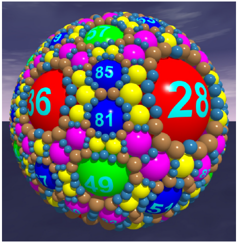


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

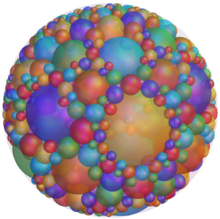


Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

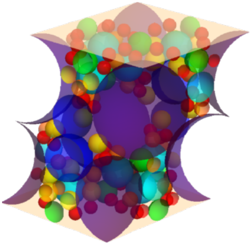


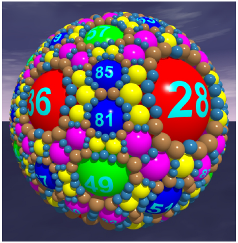
Figure: A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.

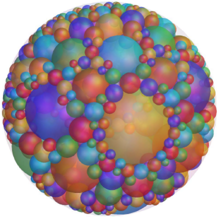
Illustrations credits

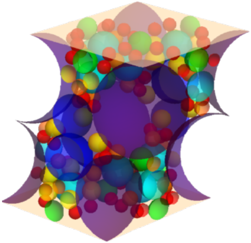
Besides the illustrations previously credited and a few Apollonian circle packing construction illustrations created by the presenter, the illustrations for this talk came from the following papers:

- Jean Bourgain and Alex Kontorovich, "On the local-global conjecture for integral Apollonian gaskets," *Inventiones mathematicae*, volume 196, pp. 589–650, 2014.
- Alex Kontorovich, "From Apollonius to Zaremba: Local-global phenomena in thin orbits," *Bulletin of the American Mathematical Society*, volume 50, number 2, pp. 187-228, 2013, `https://www.ams.org/journals/bull/2013-50-02/S0273-0979-2013-01402-2/`.
- Alex Kontorovich, "The Local-Global Principle for Integral Soddy Sphere Packings," *Journal of Modern Dynamics*, volume 15, pp. 209-236, 2019, `https://www.aimsciences.org/article/doi/10.3934/jmd.2019019`.

Thank you for listening!







Proof outline for Bourgain's and Kontorovich's Apollonian circle packing result

1. Show that the automorphism group of the Apollonian circle packing contains the congruence subgroup $\Gamma(2)$ of $PSL_2(\mathbb{Z})$, and $\Gamma(2)$ is the stabilizer of a particular circle. This implies that the set of bends contains primitive values of a shifted binary (2-variable) quadratic form. (Sarnak, 2007)

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2. The shifted binary quadratic form gives you enough to work with so that you can apply the circle method (with some other tools, including spectral theory, used in the major arc and minor arc analyses) to obtain an "almost all" statement.

Proof outline for Kontorovich's Soddy sphere packing result

1. Show that the Soddy group contains a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains "primitive" values of a shifted quaternary (4-variable) quadratic form.

Proof outline for Kontorovich's Soddy sphere packing result

1. Show that the Soddy group contains a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains "primitive" values of a shifted quaternary (4-variable) quadratic form.
2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the result of the circle method to give an asymptotic formula involving a singular series.

Proof outline for Kontorovich's Soddy sphere packing result

1. Show that the Soddy group contains a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains "primitive" values of a shifted quaternary (4-variable) quadratic form.
2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the result of the circle method to give an asymptotic formula involving a singular series.
3. Show that the singular series (with the primitivity restriction) is bounded away from zero when m is admissible.

Kleinian sphere packings

Definition (Kleinian sphere packing)

An $(n-1)$-sphere packing $\mathcal{P}$ is **Kleinian** if its limit set is that of a geometrically finite group $\Gamma < \mathrm{Isom}(\mathcal{H}^{n+1})$.

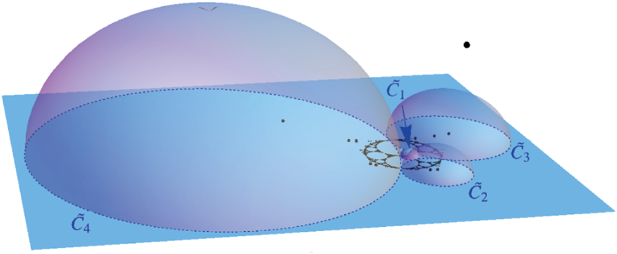

Figure: Apollonian circle packing as the limit set of Γ.

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- Action of $\mathrm{Isom}(\mathcal{H}^{n+1})$ extends continuously to $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, the boundary of $\mathcal{H}^{n+1}$.

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- Action of $\mathrm{Isom}(\mathcal{H}^{n+1})$ extends continuously to $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, the boundary of $\mathcal{H}^{n+1}$.
- Γ stabilizes $\mathcal{P}$ (i.e., Γ maps $\mathcal{P}$ to itself).