Apollonian circle packings and integers

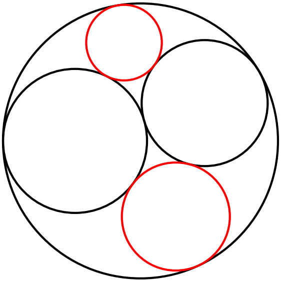
Edna Jones

Duke University

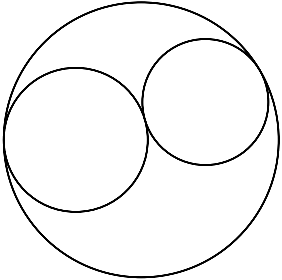
SK Day
North Carolina State University
April 13, 2024

Apollonian circle packings: The construction

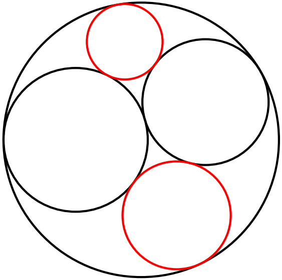
Given three pairwise tangent circles with disjoint points of tangency, there are exactly two circles tangent to the given ones. (Proved by Apollonius of Perga.)



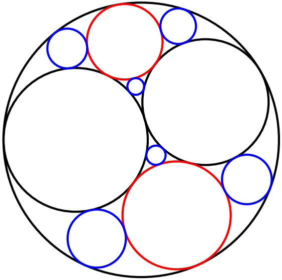
Apollonian circle packings: The construction

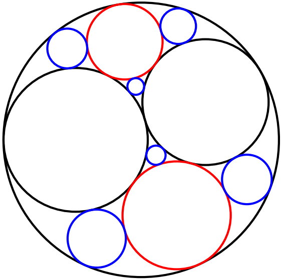


Apollonian circle packings: The construction



Apollonian circle packings: The construction





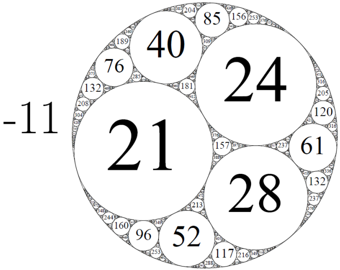


Figure: An Apollonian circle packing.

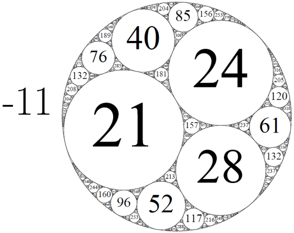


Figure: An Apollonian circle packing.

Label on circle:

bend = 1/radius

Apollonian circle packings

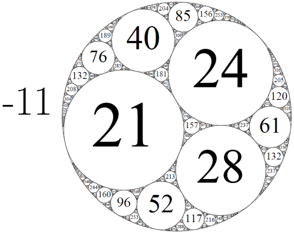


Figure: An Apollonian circle packing.

Label on circle:
bend = 1/radius

What do you notice about the bends that you can see in this Apollonian circle packing?

Apollonian circle packings

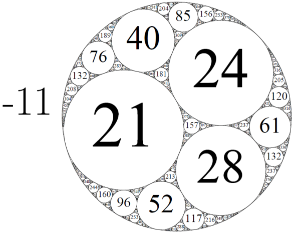


Figure: An Apollonian circle packing.

Label on circle:

bend $= 1/\text{radius}$

What do you notice about the bends that you can see in this Apollonian circle packing?

They are all integers.

Apollonian circle packings

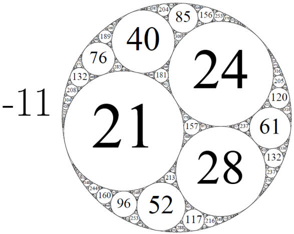


Figure: An Apollonian circle packing.

Label on circle:
bend $= 1/\text{radius}$

What do you notice about the bends that you can see in this Apollonian circle packing?

They are all integers.

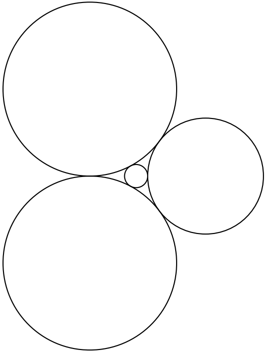
Why?

“The Kiss Precise” by F. Soddy

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.

Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

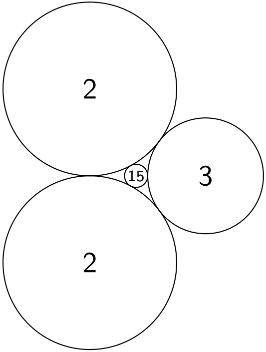
Figure: An excerpt of “The Kiss Precise” by F. Soddy in *Nature*, 1936.



Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.

Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Figure: An excerpt of "The Kiss Precise" by F. Soddy in *Nature*, 1936.



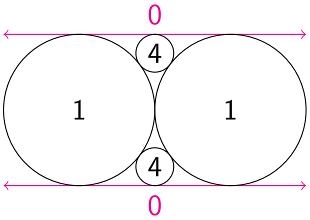
bend $= 1/\text{radius}$

“The Kiss Precise” by F. Soddy

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There’s now no need for rule of thumb.

Since zero bend’s a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Figure: An excerpt of “The Kiss Precise” by F. Soddy in *Nature*, 1936.

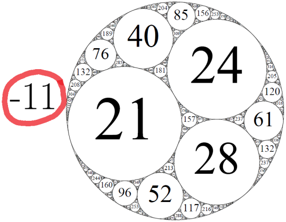


“The Kiss Precise” by F. Soddy

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There’s now no need for rule of thumb.

Since zero bend’s a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Figure: An excerpt of “The Kiss Precise” by F. Soddy in *Nature*, 1936.



"The Kiss Precise" by F. Soddy

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.

Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Figure: An excerpt of "The Kiss Precise" by F. Soddy in *Nature*, 1936.

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2.$$

Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 *are bends of four pairwise tangent circles, then*

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

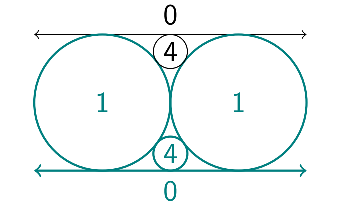
Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Example



$b_1 = 0,\ b_2 = b_3 = 1,\ b_4 = 4$

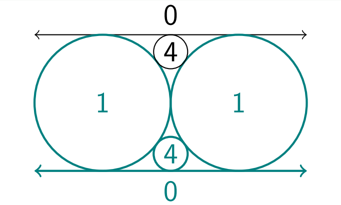
Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Example



$b_1 = 0,\ b_2 = b_3 = 1,\ b_4 = 4$

$(0 + 1 + 1 + 4)^2 = 6^2 = 36$

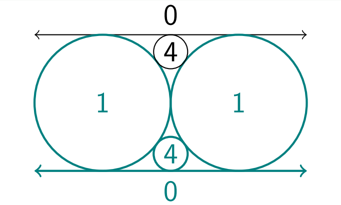
Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Example



$b_1 = 0,\ b_2 = b_3 = 1,\ b_4 = 4$

$(0 + 1 + 1 + 4)^2 = 6^2 = 36$

$2(0^2 + 1^2 + 1^2 + 4^2) = 2(18) = 36$

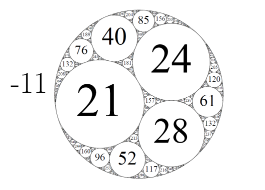
Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Example



$$b_1 = -11,\ b_2 = 21,\ b_3 = 24,\ b_4 = 28$$

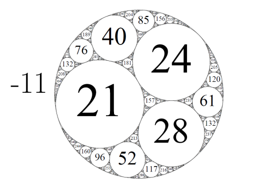
Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Example



$b_1 = -11,\ b_2 = 21,\ b_3 = 24,\ b_4 = 28$

$$(-11 + 21 + 24 + 28)^2 = 62^2 = 3844$$

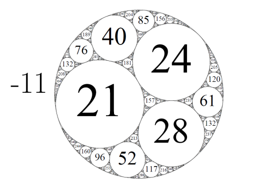
Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Example



$$b_1 = -11,\ b_2 = 21,\ b_3 = 24,\ b_4 = 28$$

$$(-11 + 21 + 24 + 28)^2 = 62^2 = 3844$$
$$2((-11)^2 + 21^2 + 24^2 + 28^2) = 2(1922) = 3844$$

Theorem (Descartes Circle Theorem, 1643)

If b_1, b_2, b_3, b_4 *are bends of four pairwise tangent circles, then*

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Fix b_1, b_2, b_3. What do we know about the solutions for b_4?

Descartes circle theorem

Theorem (Descartes Circle Theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Fix b_1, b_2, b_3. What do we know about the solutions for b_4?

We will use the quadratic formula to answer this question.

Theorem (Quadratic formula)

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Quadratic formula

Theorem (Quadratic formula)

If $ax^2 + bx + c = 0$, *then* $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example

Suppose that $2x^2 - 6x + 1 = 0$. Then

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)1}}{2(2)}$$

Quadratic formula

Theorem (Quadratic formula)

If $ax^2 + bx + c = 0$*, then* $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example

Suppose that $2x^2 - 6x + 1 = 0$. Then

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)1}}{2(2)} = \frac{6 \pm \sqrt{36 - 8}}{4}$$

Quadratic formula

Theorem (Quadratic formula)

If $ax^2 + bx + c = 0$, *then* $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example

Suppose that $2x^2 - 6x + 1 = 0$. Then

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)1}}{2(2)} = \frac{6 \pm \sqrt{36 - 8}}{4} = \frac{3}{2} \pm \frac{\sqrt{28}}{4}$$

(which can be simplified more).

Quadratic formula

Theorem (Quadratic formula)

If $ax^2 + bx + c = 0$, *then* $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example

Suppose that $2x^2 - 6x + 1 = 0$. Then

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)1}}{2(2)} = \frac{6 \pm \sqrt{36 - 8}}{4} = \frac{3}{2} \pm \frac{\sqrt{28}}{4}$$

(which can be simplified more). Thus, the solutions to $2x^2 - 6x + 1 = 0$ are

$$x = \frac{3}{2} + \frac{\sqrt{28}}{4} \qquad \text{and} \qquad x = \frac{3}{2} - \frac{\sqrt{28}}{4}.$$

The sum of the solutions to a quadratic equation

Theorem

The sum of the solutions to $ax^2 + bx + c = 0$ *is* $\dfrac{-b}{a}$.

Proof: By the quadratic formula, the sum of the solutions to $ax^2 + bx + c = 0$ is

$$\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) + \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$
$$= \frac{\left(-b+\sqrt{b^2-4ac}\right) + \left(-b-\sqrt{b^2-4ac}\right)}{2a}$$

The sum of the solutions to a quadratic equation

Theorem

The sum of the solutions to $ax^2 + bx + c = 0$ *is* $\frac{-b}{a}$.

Proof: By the quadratic formula, the sum of the solutions to $ax^2 + bx + c = 0$ is

$$\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) + \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$
$$= \frac{\left(-b+\sqrt{b^2-4ac}\right) + \left(-b-\sqrt{b^2-4ac}\right)}{2a}$$
$$= \frac{-2b}{2a}$$

The sum of the solutions to a quadratic equation

Theorem

The sum of the solutions to $ax^2 + bx + c = 0$ *is* $\dfrac{-b}{a}$.

Proof: By the quadratic formula, the sum of the solutions to $ax^2 + bx + c = 0$ is

$$\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) + \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{\left(-b+\sqrt{b^2-4ac}\right) + \left(-b-\sqrt{b^2-4ac}\right)}{2a}$$

$$= \frac{-2b}{2a} = \frac{-b}{a}.$$

The sum of the solutions to a quadratic equation

Theorem

The sum of the solutions to $ax^2 + bx + c = 0$ *is* $\dfrac{-b}{a}$.

Example

The solutions to $2x^2 - 6x + 1 = 0$ are

$$x = \frac{3}{2} + \frac{\sqrt{28}}{4} \qquad \text{and} \qquad x = \frac{3}{2} - \frac{\sqrt{28}}{4},$$

and

$$\left(\frac{3}{2} + \frac{\sqrt{28}}{4}\right) + \left(\frac{3}{2} - \frac{\sqrt{28}}{4}\right) = 3 = \frac{-(-6)}{2}.$$

Descartes circle theorem

Theorem (Descartes Circle Theorem, 1643)

If b_1, b_2, b_3, b_4 *are bends of four pairwise tangent circles, then*

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2). \qquad (*)$$

Fix b_1, b_2, b_3. What do we know about the solutions for b_4?

Descartes circle theorem

Theorem (Descartes Circle Theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2). \qquad (*)$$

Fix b_1, b_2, b_3. What do we know about the solutions for b_4?

With some algebraic manipulations, we find that Equation $(*)$ is equivalent to

$$b_4^2 - 2(b_1 + b_2 + b_3)b_4 + (b_1^2 + b_2^2 + b_3^2 - 2(b_1 b_2 + b_1 b_3 + b_2 b_3)) = 0.$$

Descartes circle theorem

Theorem (Descartes Circle Theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four pairwise tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2). \qquad (*)$$

Fix b_1, b_2, b_3. What do we know about the solutions for b_4?

With some algebraic manipulations, we find that Equation $(*)$ is equivalent to

$$b_4^2 - 2(b_1 + b_2 + b_3)b_4 + (b_1^2 + b_2^2 + b_3^2 - 2(b_1 b_2 + b_1 b_3 + b_2 b_3)) = 0.$$

Thus, if b_4 and b_4' are solutions to Equation $(*)$ for fixed b_1, b_2, b_3, then

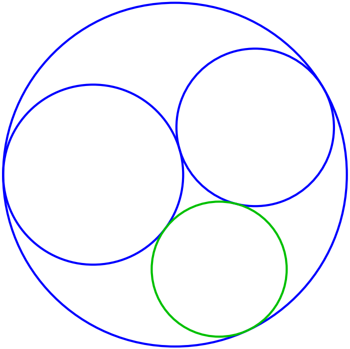
$$b_4 + b_4' = \frac{-(-2(b_1 + b_2 + b_3))}{1} = 2(b_1 + b_2 + b_3).$$

Relating to the geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$

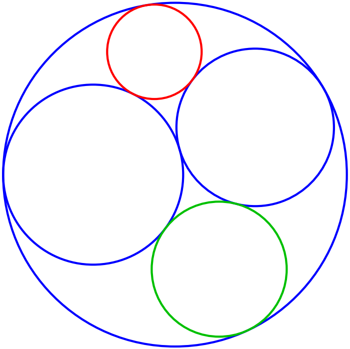
Relating to the geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



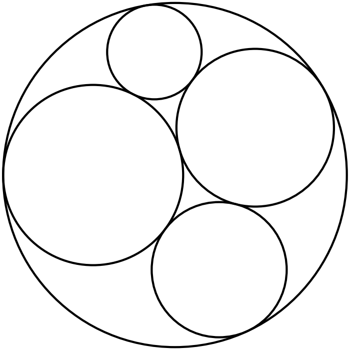
Relating to the geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



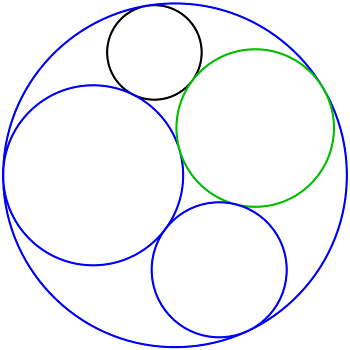
Relating to the geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



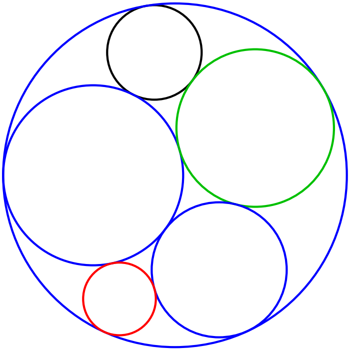
Relating to the geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



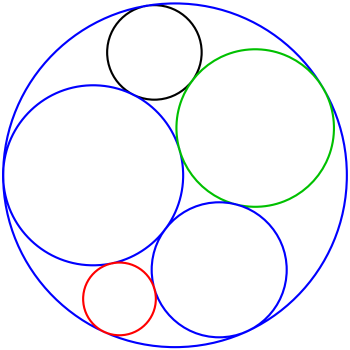
Relating to the geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



Relating to the geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$

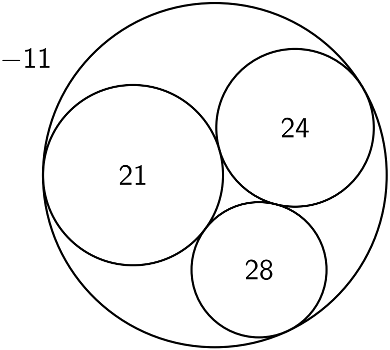


Adding, subtracting, and/or multiplying integers results in an integer.

Relating to the geometry

Adding, subtracting, and/or multiplying integers results in an integer.

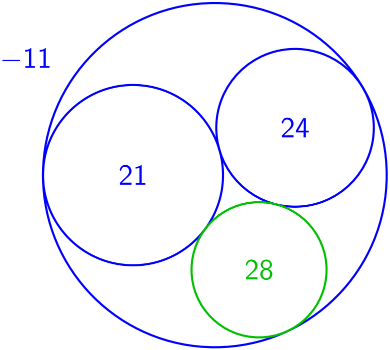
$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



Relating to the geometry

Adding, subtracting, and/or multiplying integers results in an integer.

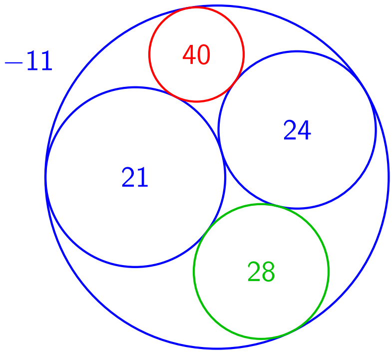
$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



Relating to the geometry

Adding, subtracting, and/or multiplying integers results in an integer.

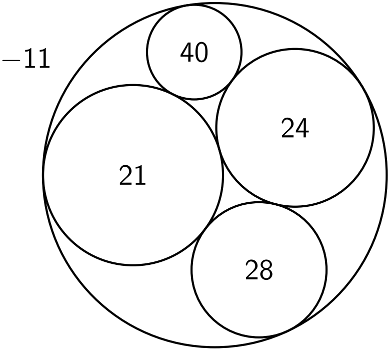
$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



Relating to the geometry

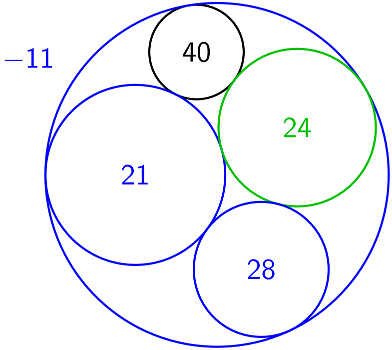
Adding, subtracting, and/or multiplying integers results in an integer.

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



Adding, subtracting, and/or multiplying integers results in an integer.

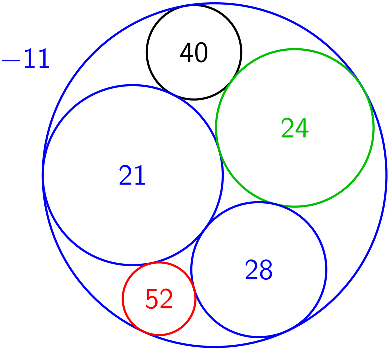
$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



Relating to the geometry

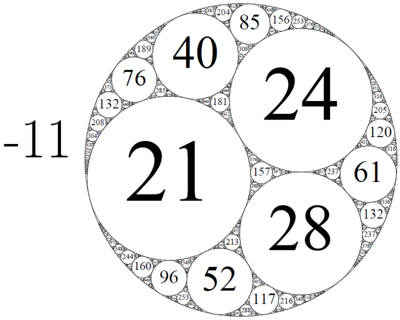
Adding, subtracting, and/or multiplying integers results in an integer.

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$



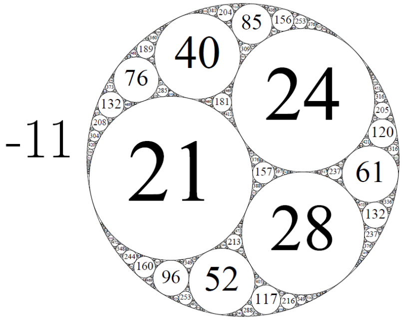
Integrality of bends

Continuing in this way, we discover that all of the bends in the following Apollonian circle packing are integers!



Integrality of bends

Continuing in this way, we discover that all of the bends in the following Apollonian circle packing are integers!



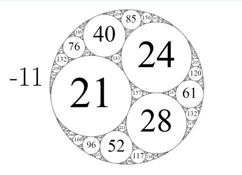
Which integers appear as bends?

Local obstructions modulo 24

Theorem (Elena Fuchs, 2011)

For an integral, primitive Apollonian circle packing, there are local obstructions modulo 24 for the bends in the packing.
That is, the remainder of a bend divided by 24 is in a certain subset of remainders.
(The local obstructions depend on the packing.)

Example



each bend
$\equiv 0, 4, 12, 13, 16, \text{or } 21 \pmod{24}.$

Admissible integers

Definition (Admissible integers for Apollonian circle packings)

Let $\mathcal{P}$ be an integral Apollonian circle packing.
An integer m is **admissible (or locally represented)** if for every integer $q \geq 1$

$$m \equiv \text{bend of some circle in } \mathcal{P} \pmod{q}.$$

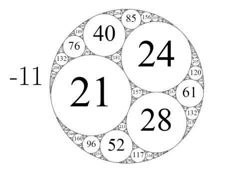
(That is, for every integer $q \geq 1$, there exists a circle in the packing with bend n such that m and n have the same remainders when dividing by q.)

Admissible integers

Theorem (Elena Fuchs, 2011)

An integer m is admissible if and only if m is in certain congruence classes modulo 24 (i.e., the remainder of m divided by 24 is in a certain subset of remainders).
(The congruence classes depend on the packing.)

Example



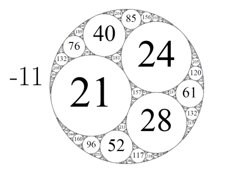
m is admissible $\iff$
$m \equiv 0, 4, 12, 13, 16, \text{or } 21 \pmod{24}$.

(Strong asymptotic) local-global conjecture

Conjecture (Ronald Graham–Jeffrey Lagarias–Colin Mallows–Allan Wilks–Catherine Yan, 2003)

The bends in a fixed primitive, integral Apollonian circle packing $\mathcal{P}$ satisfy a (strong asymptotic) local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a circle in the packing.

Example



Mathematicians thought that if
$m \equiv 0, 4, 12, 13, 16, \text{or } 21 \pmod{24}$
and m is sufficiently large,
then m is the bend of a circle in the packing.
We do not have a proof of this!

THE LOCAL-GLOBAL CONJECTURE FOR APOLLONIAN CIRCLE PACKINGS IS FALSE

SUMMER HAAG, CLYDE KERTZER, JAMES RICKARDS, AND KATHERINE E. STANGE

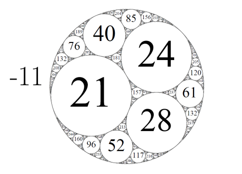
THE LOCAL-GLOBAL CONJECTURE FOR APOLLONIAN CIRCLE PACKINGS IS FALSE

SUMMER HAAG, CLYDE KERTZER, JAMES RICKARDS, AND KATHERINE E. STANGE

Theorem (Summer Haag–Clyde Kertzer–James Rickards–Katherine Stange, 2023)

Certain quadratic and quartic families of bends (of the form cn^2 and cn^4 for a fixed integer c) are missing from some Apollonian circle packings.

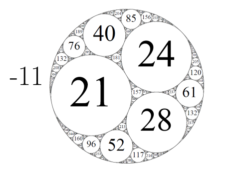
Example



An integer of the form n^2 or $3n^2$ (where n is an integer) cannot appear as the bend of a circle in this packing.

The (strong asymptotic) local-global conjecture is false

Example



An integer of the form n^2 or $3n^2$ (where n is an integer) cannot appear as the bend of a circle in this packing.

There are infinitely many admissible integers $m \equiv 0, 4, 12, 16 \pmod{24}$ that are of the form n^2 or $3n^2$.

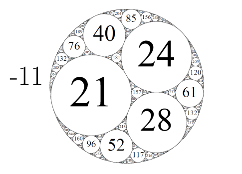
For example, there are infinitely many integers of the form $(24k)^2$ (where k is an integer) and $(24k)^2$ is divisible by 24 (so $(24k)^2 \equiv 0 \pmod{24}$).

Updated (strong asymptotic) local-global conjecture

Conjecture (Summer Haag–Clyde Kertzer–James Rickards–Katherine Stange, 2023)

There is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$, m is admissible and is not obstructed by any known quadratic or quartic obstructions, then m is the bend of a circle in the packing.

Example



It is now thought that if $m \equiv 0, 4, 12, 13, 16, \text{or } 21 \pmod{24}$, m is sufficiently large, and m is not of the form n^2 or $3n^2$, then m is the bend of a circle in the packing.

We do not have a proof of this!

Some research I am working on

Try to determine which integers are represented as bends in certain integral sphere packings in dimension at least 3.

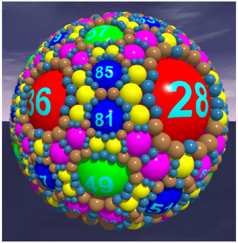


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

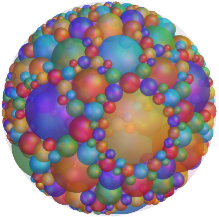


Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

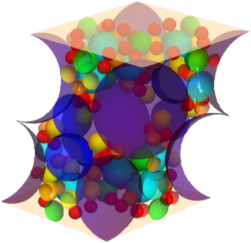


Figure: A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.

Illustrations credits

Besides the illustrations previously credited and a few Apollonian circle packing construction illustrations created by the presenter, the illustrations for this talk came from the following paper:

- Jean Bourgain and Alex Kontorovich, "On the local-global conjecture for integral Apollonian gaskets," *Inventiones mathematicae*, volume 196, pp. 589–650, 2014.

Thank you for listening!