BARD 4 Pre-Talk

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Outline

Partitions of a positive integer

Definition (Partition of a positive integer)

Let n be a positive integer. A **partition** of n is a way to write n as the sum of positive integers, where the order of the summands **does not** matter.

Example (Partitions of 5)

5	$2+2+1$
$4+1$	$2+1+1+1$
$3+2$	$1+1+1+1+1$
$3+1+1$	

Number of partitions of n

For a positive integer n, let $p(n)$ be the number of partitions of n. By convention, $p(0) = 1$.

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$\implies p(5) = 7$

Generating functions

Definition (Generating function)

The **(ordinary) generating function** for the sequence $(a_k)_{k=0}^{\infty}$ is defined to be

$$f(q) = A(q) = \sum_{k=0}^{\infty} a_k q^k.$$

Example (Generating function for $(1, 1, 1, \ldots)$)

The generating function for the sequence $(1, 1, 1, \ldots)$ is

$$1 + q + q^2 + q^3 + \cdots = \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$$

if $|q| < 1$.

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$\vdots$

- k: $q^{0\cdot k} + q^{1\cdot k} + q^{2\cdot k} + q^{3\cdot k} + \cdots = \sum_{j=0}^{\infty} (q^k)^j = \frac{1}{1-q^k}$

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Generating function for $(p(n))_{n=0}^{\infty}$

Theorem (Generating function for $(p(n))_{n=0}^{\infty}$)

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty}(q^{0\cdot k} + q^{1\cdot k} + q^{2\cdot k} + q^{3\cdot k} + \cdots)$$
$$= \prod_{k=1}^{\infty} \frac{1}{1-q^k}$$

for $|q| < 1$.

q-Pochhammer symbol

Definition (q-Pochhammer symbol)

For a nonnegative integer n,

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}).$$

Example

$$(q;q)_3 = \prod_{k=0}^{2}(1-q^{k+1}) = (1-q)(1-q^2)(1-q^3)$$

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Also, $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$.

Example

$$(q;q)_\infty = \prod_{k=0}^{\infty}(1-q^{k+1}) = \prod_{k=1}^{\infty}(1-q^k)$$

Generating function for $(p(n))_{n=0}^{\infty}$

Theorem (Generating function for $(p(n))_{n=0}^{\infty}$)

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = \frac{1}{(q;q)_\infty}$$

for $|q| < 1$.

Number of partitions of n with restrictions

For a positive integer n, let $p_d(n)$ be the number of partitions of n into distinct parts. (By convention, $p_d(0) = 1$.)
For a positive integer n, let $p_o(n)$ be the number of partitions of n into (only) odd parts. (By convention, $p_o(0) = 1$.)

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Example (Partitions of 6)

6	$3+1+1+1$
$5+1$	$2+2+2$
$4+2$	$2+2+1+1$
$4+1+1$	$2+1+1+1+1$
$3+3$	$1+1+1+1+1+1$
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Number of partitions of n with restrictions

Theorem (Euler)

The number of partitions of n into distinct parts is the same as the number of partitions of n into odd parts. That is,

$$p_d(n) = p_o(n)$$

for each nonnegative integer n.

This is called a partition identity. Finding and proving partition identities is an active area of research (see, for example, the Kanade–Russell identities).

Proof

Let $P_d(q) = \sum_{n=0}^{\infty} p_d(n)q^n$ and $P_o(q) = \sum_{n=0}^{\infty} p_o(n)q^n$.

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$$P_d(q) = \prod_{k=1}^{\infty}(1+q^k)$$

$$P_o(q) = \prod_{k=1}^{\infty}\left(\sum_{j=0}^{\infty}(q^{2k-1})^j\right) = \prod_{k=1}^{\infty}\frac{1}{1-q^{2k-1}} = \frac{1}{(q;q^2)_\infty}$$

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To prove the theorem, it suffices to show that $P_d(q) = P_o(q)$.

Proof continued

$$P_d(q) = \prod_{k=1}^{\infty}(1+q^k) = \prod_{k=1}^{\infty}\frac{1-q^{2k}}{1-q^k}$$

Proof continued

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$$= \prod_{k=1}^{\infty}\frac{1-q^{2k}}{(1-q^{2k-1})(1-q^{2k})}$$

Proof continued

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&= \prod_{k=1}^{\infty}\frac{1}{1-q^{2k-1}} \\
&= P_o(q)
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$$

$\implies p_d(n) = p_o(n)$ for each nonnegative integer n. □

Thank you for listening so far!

(We are taking a break before discussing the circle method.)

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  - Kloosterman circle method
  - The delta(-symbol) method
- Wright's circle method is used to give asymptotic formulas for Fourier coefficients of analytic functions

Hardy–Littlewood circle method

- Originally developed by Hardy and Ramanujan (1918) to provide asymptotic formula for the partition function $p(n)$, the number of partitions of n
- Proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

Partition function & modularity

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz)\,,$$

where $\mathrm{e}(z) = e^{2\pi iz}$ and $\mathrm{Im}(z) > 0$.

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$$f(z) = \frac{\mathrm{e}(z/24)}{\eta(z)},$$

where $\eta(z)$ is the Dedekind eta function

$$\eta(z) = \mathrm{e}\left(\frac{z}{24}\right) \prod_{m=1}^{\infty} (1 - \mathrm{e}(mz)).$$

Hardy and Ramanujan used the modularity of η to obtain the asymptotic formula.

Hardy–Littlewood circle method & partition function

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz) \qquad F(q) = \sum_{n=0}^{\infty} p(n)q^n$$

where $q = \mathrm{e}(z)$.

Hardy–Littlewood circle method & partition function

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Using the Cauchy integral formula, we find that

$$p(n) = \frac{1}{2\pi i} \int_{|q|=r} \frac{F(q)}{q^{n+1}}\, dq,$$

where $0 < r < 1$.

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where $0 < r < 1$.
Changing q into $\mathrm{e}(x + iy)$, we obtain

$$p(n) = \int_0^1 f(x + iy)\mathrm{e}(-n(x + iy))\, dx,$$

where $y > 0$ is such that $r = e^{-2\pi y}$.

Partition function

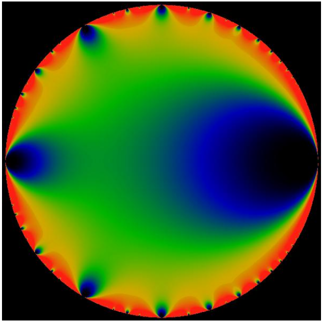

Figure: Modulus of $\prod_{m=1}^{\infty}(1 - q^m)$ with $|q| < 1$. From Wikipedia.

Main contribution to integral from points near $\mathrm{e}(a/q)$ where q is small.

Major arcs and minor arcs

Split $[0, 1]$ into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$.

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$$\mathfrak{M} = \left\{ x \in [0, 1] : x \text{ is ``close to''} \frac{a}{q}, a, q \in \mathbb{Z}, 0 < q \leq Q \right\}.$$

How close depends on the application of the method.

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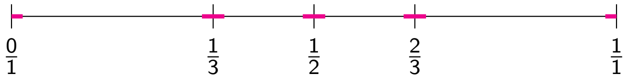
How close depends on the application of the method.

$$\mathfrak{m} = [0,1] \setminus \mathfrak{M}.$$

Example of major arcs $\mathfrak{M}$ when $Q = 3$ for the Hardy–Littlewood circle method:



$$
\begin{aligned}
p(n) &= \int_0^1 f(x+iy)\mathrm{e}(-n(x+iy))\ dx \\
&= \underbrace{\int_{\mathfrak{M}} f(x+iy)\mathrm{e}(-n(x+iy))\ dx}_{\text{main term}} + \underbrace{\int_{\mathfrak{m}} f(x+iy)\mathrm{e}(-n(x+iy))\ dx}_{\text{error term}}
\end{aligned}
$$

Real quadratic forms

F is a real quadratic form in s variables $\iff$
For all $\mathbf{m} \in \mathbb{R}^s$,

$$F(\mathbf{m}) = \frac{1}{2}\mathbf{m}^\top A\mathbf{m},$$

where A is a real symmetric $s \times s$ matrix and is the Hessian matrix of F.

Example (Example of a quadratic form in 2 variables)

$$\begin{aligned} F(\mathbf{m}) &= m_1^2 + m_1 m_2 + m_2^2 \\ &= \frac{1}{2}\mathbf{m}^\top \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{m} \end{aligned}$$

Quadratic form definitions

Definition (Integral quadratic form)

A quadratic form F is **integral** if $F(\mathbf{m}) \in \mathbb{Z}$ for all $\mathbf{m} \in \mathbb{Z}^s$.

Definition (Positive definite quadratic form)

A quadratic form F is **positive definite** if $F(\mathbf{m}) > 0$ for all $\mathbf{m} \in \mathbb{R}^s \setminus \{\mathbf{0}\}$.

Examples (Examples of integral positive definite quadratic forms)

- $f_4(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2 + m_4^2$
- $x^2 + xy + y^2$

Hardy–Littlewood circle method & quadratic forms

Definition (Representation number)

$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

Want an asymptotic formula for $R_F(n)$ when F is a positive definite quadratic form.

Hardy–Littlewood circle method & quadratic forms

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$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

Want an asymptotic formula for $R_F(n)$ when F is a positive definite quadratic form.

Use same overall method for obtaining an asymptotic formula for the partition function.

Note that the theta function

$$\Theta(z) = \sum_{n=0}^{\infty} R_F(n)\mathrm{e}(nz)$$

is a modular form.

Singular series $\mathfrak{S}_F(n)$

$$\mathfrak{S}_F(n) = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q} \left(F(\mathbf{h}) - n \right) \right)$$

The singular series $\mathfrak{S}_F(n)$ contains information about $F(\mathbf{m}) \equiv n \pmod{q}$ for all positive integers q.

There exists a positive integer q such that $F(\mathbf{m}) \equiv n \pmod{q}$ has no solutions
$\implies \mathfrak{S}_F(n) = 0$

An asymptotic for representation numbers from Hardy–Littlewood circle method

Theorem (Kloosterman, 1924)

Suppose that n is a positive integer.
Suppose that F is a positive definite integral quadratic form in $s \geq 5$ variables.
Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F.
Then the number of integral solutions to $F(\mathbf{m}) = n$ is

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon}\right)$$

for any $\varepsilon > 0$.

Motivation for the Kloosterman circle method

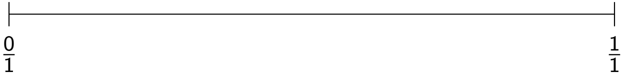
- Want a better error term in asymptotic formula for $R_F(n)$ when F is a positive definite quadratic form.
- Split $[0, 1]$ differently.

Farey sequence $\mathfrak{F}_Q$ of order Q

Definition

For $Q \geq 1$, the **Farey sequence** $\mathfrak{F}_Q$ **of order** Q is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 1$



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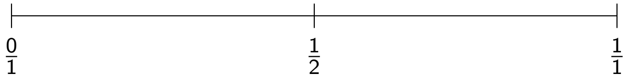
$Q = 2$



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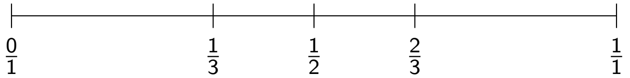
$Q = 3$



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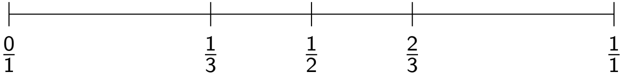
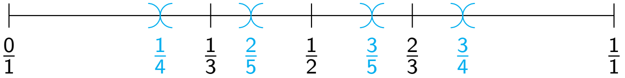


Example of Farey dissection when $Q = 3$:



An asymptotic for representation numbers from the Kloosterman method

Theorem

Suppose that n is a positive integer.
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Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F.
Then the number of integral solutions to $F(\mathbf{m}) = n$ is

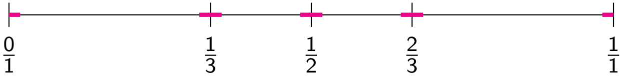
$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right)$$

for any $\varepsilon > 0$.

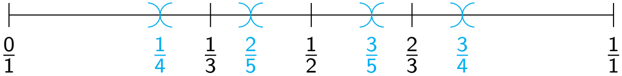
Kloosterman proved this (with a worse error term) in 1926 for diagonal quadratic forms ($F(\mathbf{m}) = a_1 m_1^2 + \cdots + a_s m_s^2$), using what is now called the Kloosterman circle method.

Hardy–Littlewood vs. Kloosterman

Example of major arcs when $Q = 3$ for the Hardy–Littlewood circle method:



Example of Farey dissection when $Q = 3$ for the Kloosterman circle method:



Hardy–Littlewood vs. Kloosterman

Hardy–Littlewood:

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon}\right)$$

Kloosterman:

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right)$$

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- Turned into a theorem by Ngo and Rhoades in 2017
    - Have two analytic functions $L(q)$ and $\xi(q)$ on region defined by $|q| < 1$ and $q \notin \mathbb{R}_{\leq 0}$
    - Need to have certain asymptotics for $L(q)$ and $\xi(q)$ as $q \to 1$ in a certain region
    - Outside that region, need $|L(q)|$ and $|\xi(q)|$ satisfying certain upper bounds
    - Conclusion: Asymptotic formula for $a(n)$, where $L(q)\xi(q) = \sum_{n=0}^{\infty} a(n)q^n$

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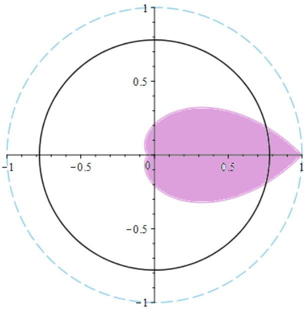
  - Conclusion: Asymptotic formula for $a(n)$, where $L(q)\xi(q) = \sum_{n=0}^{\infty} a(n)q^n$
- Used in a variety of partition-counting and q-series applications, including
  - Counting the number of parts in all partitions of n that are in any given arithmetic progression (Beckwith–Mertens)
  - Oscillating asymptotics for coefficients of a Nahm-type sum (Folsom–Males–Rolen–Storzer)

Example of circle of integration for Wright's circle method:



If certain analytic functions satisfy certain conditions, then we can obtain an asymptotic formula for the coefficients of the associated generating function.

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- More versatile than Kloosterman circle method
- Developed by Duke, Friedlander, and Iwaniec in 1993 to compute bounds for automorphic L-functions
- Has been used for a variety of applications, including
    - Asymptotic formulas for the representation numbers of quadratic forms (Heath-Brown)
    - Subconvexity bounds for (twists of) automorphic forms (Munshi)

Representation number

Definition (Representation number)

$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}},$$

where $\mathbf{1}_{\{F(\mathbf{m})=n\}}$ is the indicator function

$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \begin{cases} 1 & \text{if } F(\mathbf{m}) = n, \\ 0 & \text{otherwise.} \end{cases}$$

Indicator function

$$\delta(n) = \mathbf{1}_{\{n=0\}} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

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The delta method & bump functions

Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}$ is denoted by $C_c^\infty(\mathbb{R})$. A function $w \in C_c^\infty(\mathbb{R})$ is called a **bump function**.

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If n is an integer, then

$$\delta(n) = \frac{1}{\sum_{q=1}^\infty w(q)} \sum_{q|n} \left(w(q) - w\left(\frac{|n|}{q}\right)\right),$$

where the sum over $q \mid n$ is taken to be the sum over the positive divisors of n.

The delta method & bump functions

Using the fact that

$$\frac{1}{q} \sum_{a \pmod{q}} \mathrm{e}\left(\frac{an}{q}\right) = \begin{cases} 1 & \text{if } q \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

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we have

$$\begin{aligned} \delta(n) &= \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q \mid n} \left(w(q) - w\left(\frac{|n|}{q}\right)\right) \\ &= \frac{1}{\sum_{q=1}^{\infty} w(q)} \sum_{q=1}^{\infty} \frac{1}{q} \sum_{a \pmod q} \mathrm{e}\left(\frac{an}{q}\right) \left(w(q) - w\left(\frac{|n|}{q}\right)\right) \end{aligned}$$

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The delta method

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Bump functions are easier to handle analytically than the discontinuous delta function, which helps when analyzing

$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \delta(F(\mathbf{m}) - n).$$

Specifics depend on the application of the delta method.

Some (of the many) applications of the circle method

- Asymptotic formula for the partition function (Hardy–Ramanujan)

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- Subconvexity bounds for (twists of) automorphic forms (Duke–Friedlander–Iwaniec, Munshi)
- A variety of partition-counting and q-series applications (Beckwith–Mertens, Folsom–Males–Rolen–Storzer)

Thank you for listening!

Lemma for Kloosterman circle method

Lemma

Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic function of period 1 and with real Fourier coefficients (so that $\overline{f(x)} = f(-x)$ for all $x \in \mathbb{R}$). Then

$$\int_0^1 f(x)\, dx = 2\,\mathrm{Re}\left(\sum_{1 \le q \le Q} \int_0^{\frac{1}{qQ}} \sum_{\substack{Q < d \le q+Q \\ qdx < 1 \\ \gcd(d,q)=1}} f\left(x - \frac{d^*}{q}\right)\, dx \right),$$

where d^* is the multiplicative inverse of d modulo q.

Use this for

$$f(x) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathrm{e}((x+iy)(F(\mathbf{m}) - n)),$$

where $y > 0$.