# The circle method

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**Definition (Representation number)**

$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

# Hardy-Littlewood circle method

- Originally developed by Hardy and Ramanujan (1918) to provide asymptotic formula for the partition function $p(n)$, the number of partitions of $n$
- Proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

# Partition function & modularity

$$f(z) = \sum_{n=0}^{\infty} p(n)\mathrm{e}(nz)\,,$$

where $\mathrm{e}(z) = e^{2\pi iz}$ and $\mathrm{Im}(z) > 0$.

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where $\mathrm{e}(z) = e^{2\pi iz}$ and $\mathrm{Im}(z) > 0$.

$$f(z) = \frac{\mathrm{e}(z/24)}{\eta(z)},$$

where $\eta(z)$ is the Dedekind eta function

$$\eta(z) = \mathrm{e}\left(\frac{z}{24}\right) \prod_{m=1}^{\infty} (1 - \mathrm{e}(mz)).$$

Hardy and Ramanujan used the modularity of $\eta$ to obtain the asymptotic formula.

# Hardy-Littlewood circle method & partition function

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$$p(n) = \frac{1}{2\pi i} \int_{|q|=r} \frac{F(q)}{q^{n+1}}\, dq,$$

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where $0 < r < 1$.
Changing $q$ into $\mathrm{e}(x + iy)$, we obtain

$$p(n) = \int_0^1 f(x + iy)\mathrm{e}(-n(x + iy))\, dx,$$

where $y > 0$.

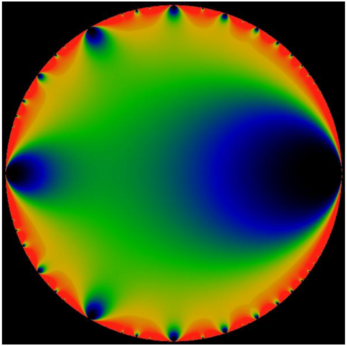

Figure: Modulus of $\prod_{m=1}^{\infty}(1 - q^m)$ with $|q| < 1$. From Wikipedia.

Main contribution to integral from points near $\mathrm{e}(a/q)$ where $q$ is small.

# Major arcs and minor arcs

Split $[0, 1]$ into major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$.

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$$\mathfrak{M} = \left\{ x \in [0,1] : x \text{ is ``close to''} \frac{a}{q}, a, q \in \mathbb{Z}, 0 < q \leq Q \right\}.$$

How close depends on the application of the method.

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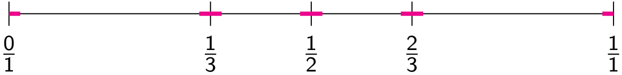
$$\mathfrak{m} = [0,1] \setminus \mathfrak{M}.$$

Example of major arcs $\mathfrak{M}$ when $Q = 3$ for the Hardy-Littlewood circle method:




$$p(n) = \int_0^1 f(x+iy)\mathrm{e}(-n(x+iy))\ dx$$
$$= \int_{\mathfrak{M}} f(x+iy)\mathrm{e}(-n(x+iy))\ dx + \int_{\mathfrak{m}} f(x+iy)\mathrm{e}(-n(x+iy))\ dx$$

# Real quadratic forms

$F$ is a real quadratic form in $s$ variables $\iff$
For all $\mathbf{m} \in \mathbb{R}^s$,

$$F(\mathbf{m}) = \frac{1}{2}\mathbf{m}^\top A\mathbf{m},$$

where $A$ is a real symmetric $s \times s$ matrix and is the Hessian matrix of $F$.

**Example (Example of a quadratic form in 2 variables)**

$$\begin{aligned} F(\mathbf{m}) &= m_1^2 + m_1 m_2 + m_2^2 \\ &= \frac{1}{2}\mathbf{m}^\top \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{m} \end{aligned}$$

# Quadratic form definitions

**Definition (Integral quadratic form)**

A quadratic form $F$ is **integral** if $F(\mathbf{m}) \in \mathbb{Z}$ for all $\mathbf{m} \in \mathbb{Z}^s$.

**Definition (Positive definite quadratic form)**

A quadratic form $F$ is **positive definite** if $F(\mathbf{m}) > 0$ for all $\mathbf{m} \in \mathbb{R}^s \setminus \{\mathbf{0}\}$.

**Examples (Examples of integral positive definite quadratic forms)**

- $f_4(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2 + m_4^2$
- $x^2 + xy + y^2$

# Hardy-Littlewood circle method & quadratic forms

Want an asymptotic formula for $R_F(n)$ when $F$ is a positive definite quadratic form.

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Use same overall method for obtaining an asymptotic formula for the partition function.

Note that the theta function

$$\Theta(z) = \sum_{n=0}^{\infty} R_F(n)\mathrm{e}(nz)$$

is an automorphic form.

# Singular series $\mathfrak{S}_F(n)$

$$\mathfrak{S}_F(n) = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left( \frac{d}{q} (F(\mathbf{h}) - n) \right)$$

The singular series $\mathfrak{S}_F(n)$ contains information about $F(\mathbf{m}) \equiv n \pmod{q}$ for all positive integers $q$.

$\mathfrak{S}_F(n) = 0 \iff$
there exists a positive integer $q$ such that $F(\mathbf{m}) \equiv n \pmod{q}$ has no solutions

# An asymptotic for representation numbers from Hardy-Littlewood circle method

**Theorem (Kloosterman, 1924)**

*Suppose that $n$ is a positive integer.*
*Suppose that $F$ is a positive definite integral quadratic form in $s \geq 5$ variables.*
*Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of $F$.*
*Then the number of integral solutions to $F(\mathbf{m}) = n$ is*

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s}{4}+\varepsilon} + n^{\frac{s}{2}-\frac{5}{4}+\varepsilon}\right)$$

*for any $\varepsilon > 0$.*

# Motivation for the Kloosterman circle method

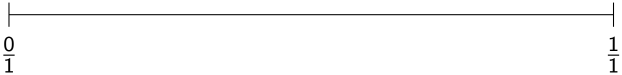
- Want a better error term in asymptotic formula for $R_F(n)$ when $F$ is a positive definite quadratic form.
- Split $[0,1]$ differently.

# Farey sequence $\mathfrak{F}_Q$ of order $Q$

## Definition

For $Q \geq 1$, the **Farey sequence** $\mathfrak{F}_Q$ **of order** $Q$ is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 1$



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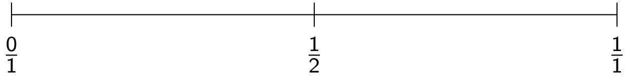
$Q = 2$



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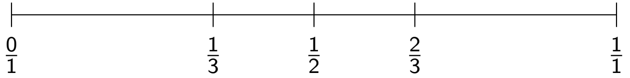

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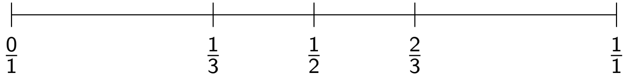
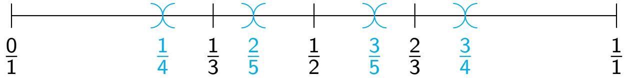



Example of Farey dissection when $Q = 3$:



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*Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of $F$.*
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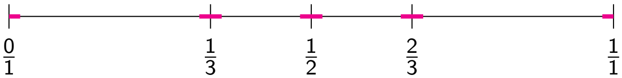
*for any $\varepsilon > 0$.*

Kloosterman proved this (with a worse error term) in 1926 for diagonal quadratic forms ($F(\mathbf{m}) = a_1 m_1^2 + \cdots + a_s m_s^2$), using what is now called the Kloosterman circle method.

# Hardy-Littlewood vs. Kloosterman

Example of major arcs when $Q = 3$ for the Hardy-Littlewood circle method:

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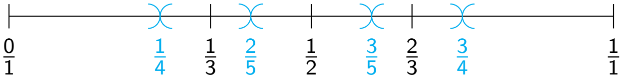

Hardy-Littlewood:

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- Rewrite $\delta(n)$, the indicator function for zero, using bump functions
- More versatile than Kloosterman circle method
- Has been used for a variety of applications, including
  - Asymptotic formulas for the representation numbers of quadratic forms (Heath-Brown)
  - Subconvexity bounds for (twists of) automorphic forms (Munshi)

# Representation number

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$$R_F(n) = \#\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}$$

$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}},$$

where $\mathbf{1}_{\{F(\mathbf{m})=n\}}$ is the indicator function

$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \begin{cases} 1 & \text{if } F(\mathbf{m}) = n, \\ 0 & \text{otherwise.} \end{cases}$$

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$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \delta(F(\mathbf{m}) - n)$$

$$R_F(n) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \delta(F(\mathbf{m}) - n)$$

**Definition (Bump function)**

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}$ is denoted by $C_c^\infty(\mathbb{R})$. A function $w \in C_c^\infty(\mathbb{R})$ is called a **bump function**.

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If $n$ is an integer, then

$$\delta(n) = \frac{1}{\sum_{q=1}^\infty w(q)} \sum_{q \mid n} \left( w(q) - w\left( \frac{|n|}{q} \right) \right),$$

where the sum over $q \mid n$ is taken to be the sum over the positive divisors of $n$.

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- Subconvexity bounds for (twists of) automorphic forms (Munshi)

Thank you for listening!

# Lemma for Kloosterman circle method

## Lemma

*Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic function of period 1 and with real Fourier coefficients (so that $\overline{f(x)} = f(-x)$ for all $x \in \mathbb{R}$). Then*

$$\int_0^1 f(x)\,dx = 2\,\mathrm{Re}\left( \sum_{1 \le q \le Q} \int_0^{\frac{1}{qQ}} \sum_{\substack{Q < d \le q+Q \\ qdx < 1 \\ \gcd(d,q)=1}} f\left(x - \frac{d^*}{q}\right)\,dx \right),$$

*where $d^*$ is the multiplicative inverse of $d$ modulo $q$.*

Use this for

$$f(x) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathrm{e}((x + iy)(F(\mathbf{m}) - n)),$$

where $y > 0$.