The Descartes circle theorem

How kissing circles give rise to a quadratic equation

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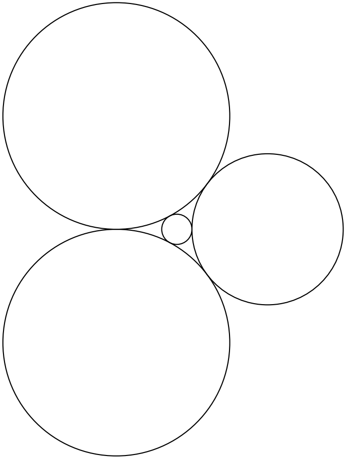
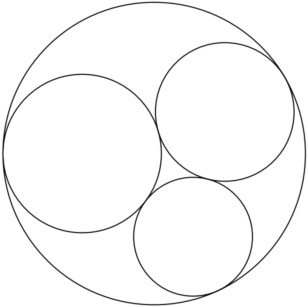
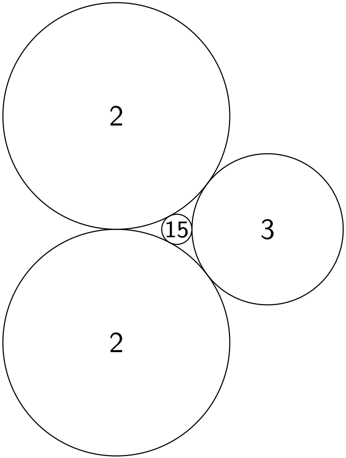
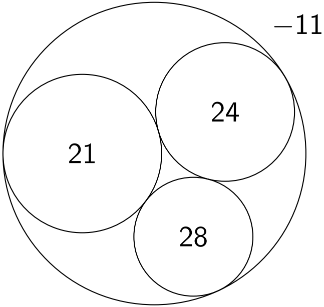
Kissing (mutually tangent) circles





curvature = bend = 1/radius

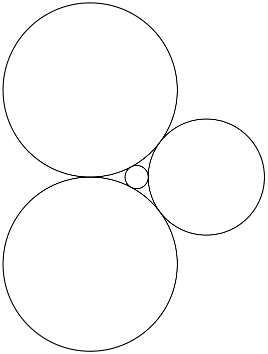




Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There's now no need for rule of thumb.

Since zero bend's a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Figure: An excerpt of "The Kiss Precise" by F. Soddy in *Nature*, 1936.



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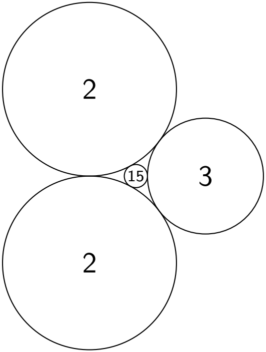

bend $=$ 1/radius

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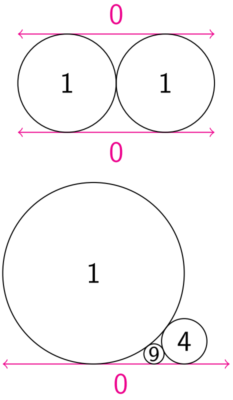

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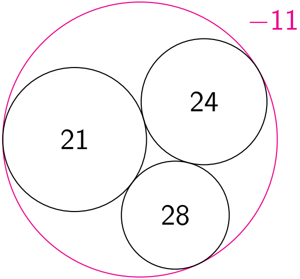

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If b_1, b_2, b_3, b_4 are bends of four mutually tangent circles, then

$$\sum_{j=1}^{4} {b_j}^2 = \frac{1}{2}\left(\sum_{j=1}^{4} b_j\right)^2 .$$

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four mutually tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Descartes circle theorem

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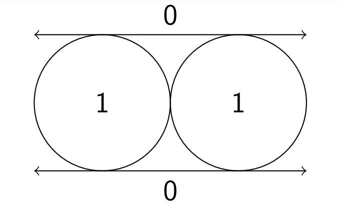

$$b_1 = b_2 = 0,\ b_3 = b_4 = 1$$

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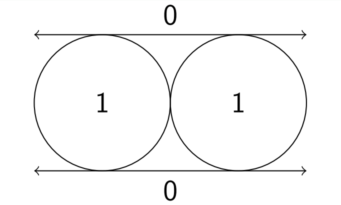

$$b_1 = b_2 = 0, \; b_3 = b_4 = 1$$

$$(0 + 0 + 1 + 1)^2 = 2^2 = 4$$

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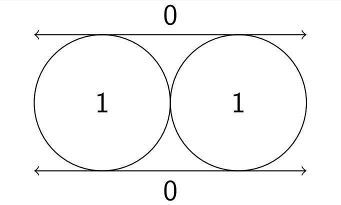

$b_1 = b_2 = 0, \; b_3 = b_4 = 1$

$(0 + 0 + 1 + 1)^2 = 2^2 = 4$

$2(0^2 + 0^2 + 1^2 + 1^2) = 2(2) = 4$

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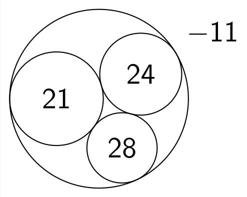

$$b_1 = -11,\ b_2 = 21,\ b_3 = 24,\ b_4 = 28$$

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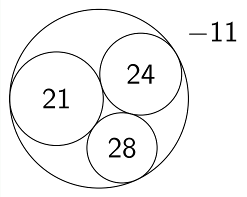

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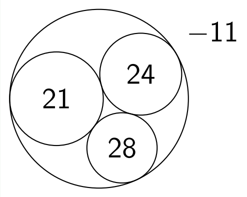


$$b_1 = -11,\ b_2 = 21,\ b_3 = 24,\ b_4 = 28$$

$$(-11 + 21 + 24 + 28)^2 = 62^2 = 3844$$
$$2((-11)^2 + 21^2 + 24^2 + 28^2) = 2(1922) = 3844$$

An incomplete history of the Descartes circle theorem

- 1643 – René Descartes wrote the theorem and an incomplete proof of it in a letter to Princess Elisabeth of Bohemia.
- 1826 – Jakob Steiner independently rediscovered the theorem and provided a complete proof of it.
- 1842 – Philip Beecroft independently rediscovered the theorem and provided a complete proof of it.
- 1936 – Frederick Soddy published "The Kiss Precise."
- 1967 – Daniel Pedoe called the theorem the Descartes circle theorem and published multiple (not all original) proofs of it.
- 1968 – H.S.M. Coxeter published a proof of the theorem.

Integral Apollonian circle packings

The Descartes circle theorem can be used to show that the bend of each circle in this Apollonian circle packing is an integer!

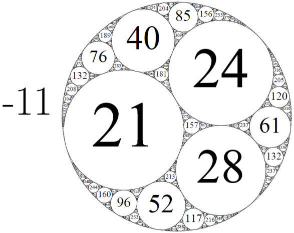


Figure: An integral Apollonian circle packing from "On the local-global conjecture for integral Apollonian gaskets" by Jean Bourgain and Alex Kontorovich.

Integral Apollonian circle packings

The Descartes circle theorem can be used to show that the bend of each circle in this Apollonian circle packing is an integer!

Figuring out which integers can appear as bends in a particular Apollonian circle packing is an open problem.

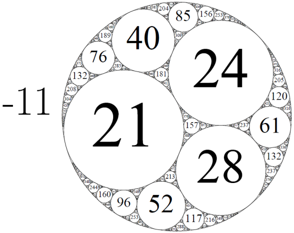
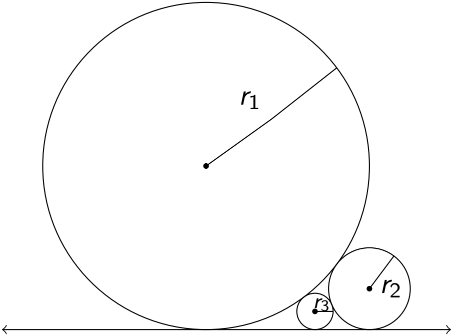


Figure: An integral Apollonian circle packing from "On the local-global conjecture for integral Apollonian gaskets" by Jean Bourgain and Alex Kontorovich.

What is the relationship between the radii of three circles of different sizes all tangent to the same line and each externally tangent to the other two?



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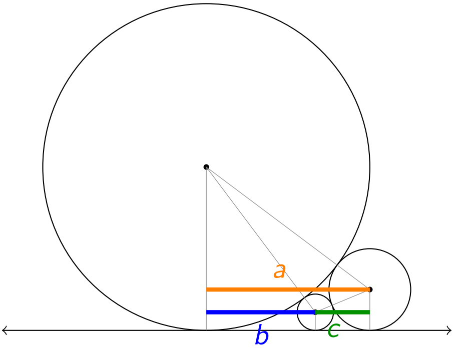

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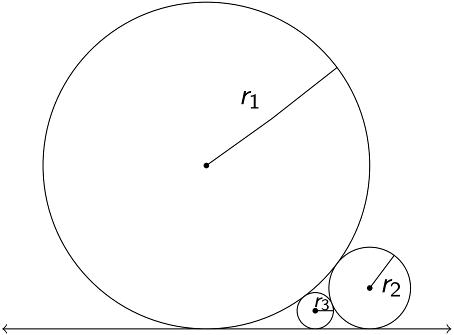

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

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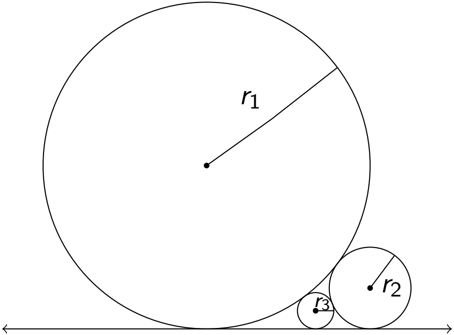

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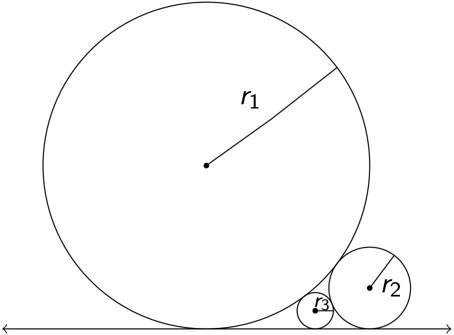


$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

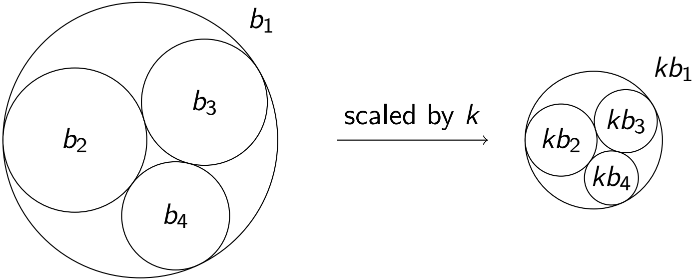
$$\iff \sqrt{b_3} = \sqrt{b_1} + \sqrt{b_2}$$

$$\implies$$

$$(b_1 + b_2 + b_3)^2 = 2(b_1^2 + b_2^2 + b_3^2),$$

which is what we want since $b_4 = 0$.

Equation holds under scaling



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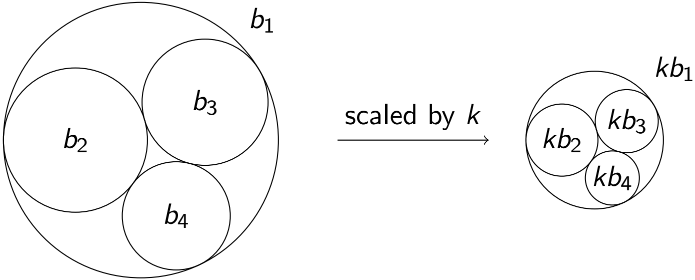


If the Descartes circle theorem is true, then

$$\left(\sum_{j=1}^{4} kb_j\right)^2 = k^2\left(\sum_{j=1}^{4} b_j\right)^2 = k^2 \cdot 2\left(\sum_{j=1}^{4} {b_j}^2\right) = 2\left(\sum_{j=1}^{4} (kb_j)^2\right). \checkmark$$

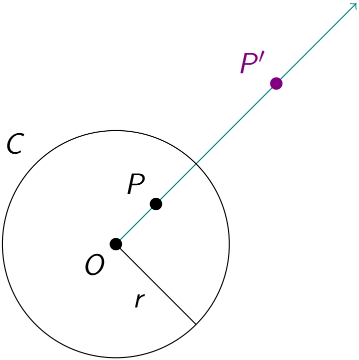
A proof by Coxeter

- Uses the fact that

$$\left(\sum_{j=1}^{4} b_j\right)^2 = 2\sum_{j=1}^{4} {b_j}^2$$

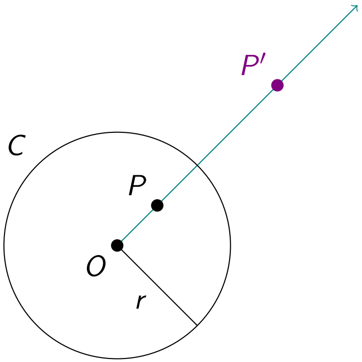
  holds for mutually tangent circles under scaling, translation, and rotation
- Uses circle inversion

Circle inversion



P' = the inversion of Point P in Circle C.

$$OP \cdot OP' = r^2$$



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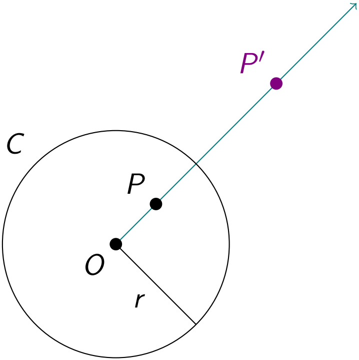

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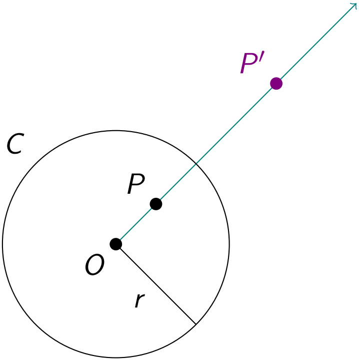
Observation: $(P')' = P$



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$$\Longrightarrow \quad OP' = \frac{r^2}{OP}$$

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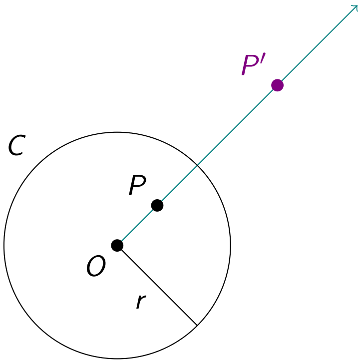
Special case: $O' = ?$



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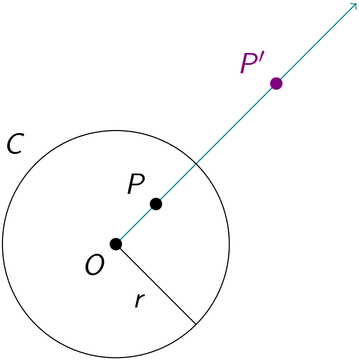

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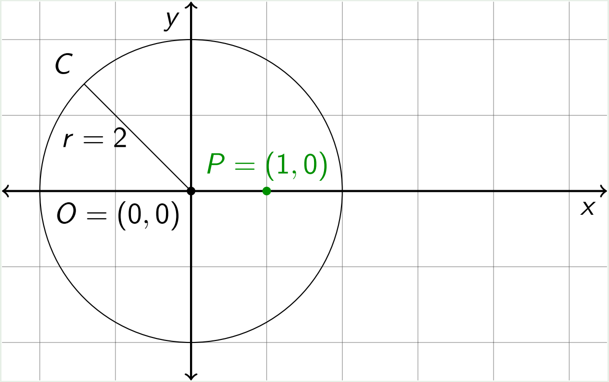
Circle inversion example

Example



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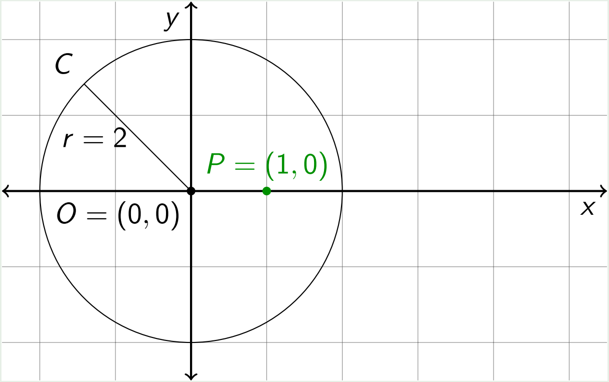

$$OP' = \frac{r^2}{OP} = \frac{2^2}{1} = 4$$

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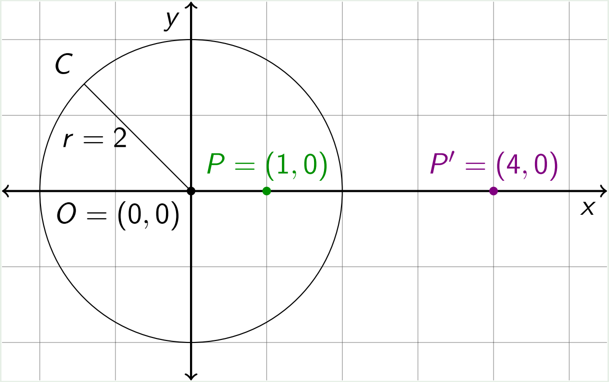

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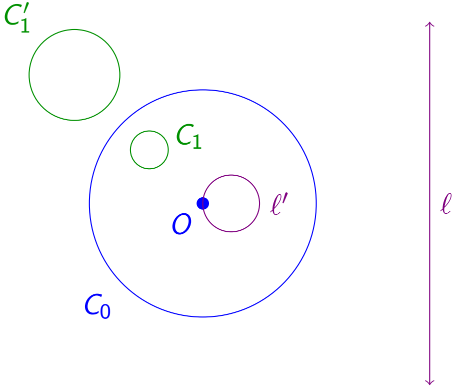
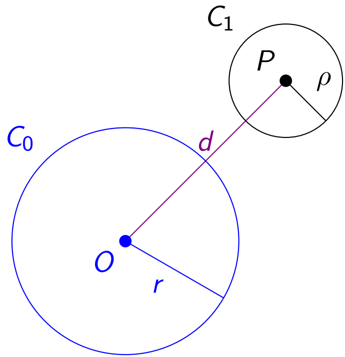
Circle inversion on circles and lines

Circle inversion sends generalized circles to generalized circles.

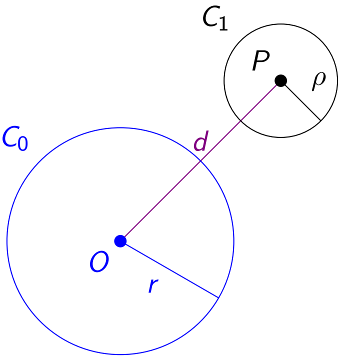
Generalized circle = circle or line





Invert Circle C_1 in Circle C_0.

Inverting a circle

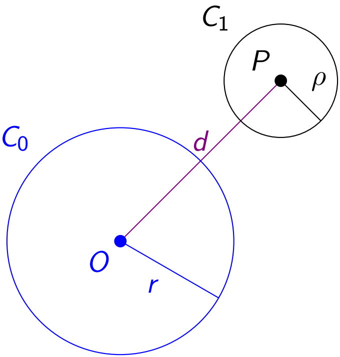


Invert Circle C_1 in Circle C_0.

Inverted circle has

- (signed) radius $= \dfrac{r^2\rho}{d^2 - \rho^2}$

Inverting a circle



Invert Circle C_1 in Circle C_0.

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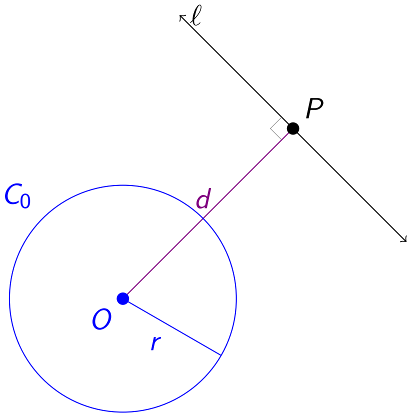
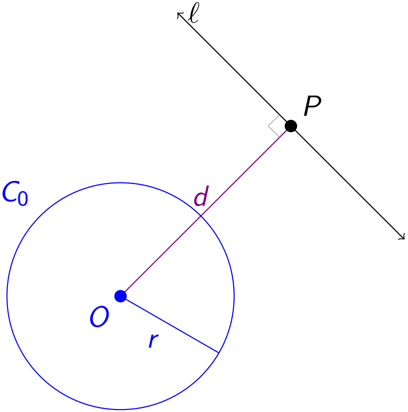
- bend $= \dfrac{d^2 - \rho^2}{r^2\rho}$



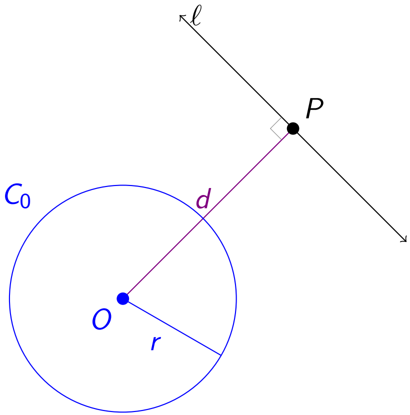
Invert Line ℓ in Circle C_0.



Invert Line ℓ in Circle C_0.

Resulting circle has

- (signed) radius $= \pm\dfrac{r^2}{2d}$



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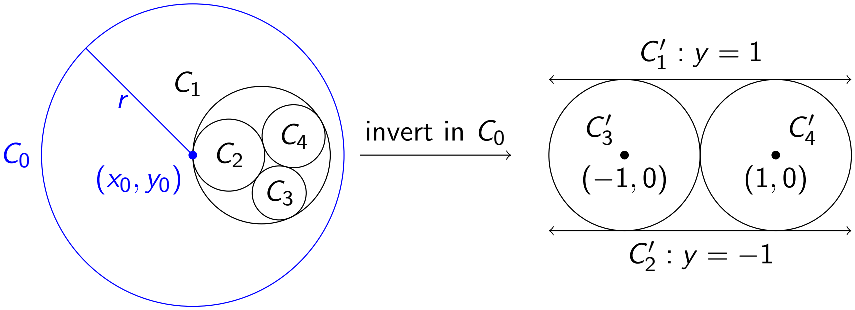
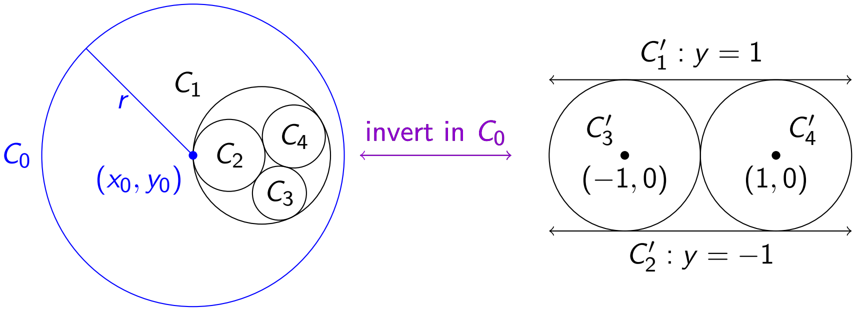
- (signed) radius $= \pm\frac{r^2}{2d}$
- bend $= \pm\frac{2d}{r^2}$

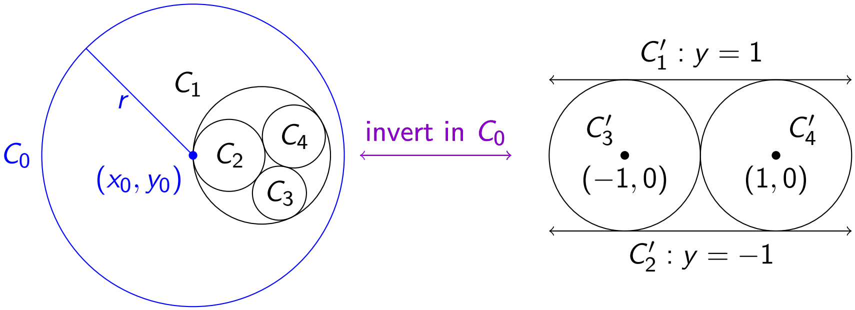


(up to scaling, translation, and rotation)



(up to scaling, translation, and rotation)

Coxeter's proof



(up to scaling, translation, and rotation)

$$b_1 = \frac{2(1 - y_0)}{r^2} \qquad b_2 = \frac{2(1 + y_0)}{r^2}$$

$$b_3 = \frac{{x_0}^2 + {y_0}^2 + 2x_0}{r^2} \qquad b_4 = \frac{{x_0}^2 + {y_0}^2 - 2x_0}{r^2}$$

Coxeter's proof

$$b_1 = \frac{2(1-y_0)}{r^2}, \qquad b_2 = \frac{2(1+y_0)}{r^2},$$
$$b_3 = \frac{{x_0}^2 + {y_0}^2 + 2x_0}{r^2}, \qquad b_4 = \frac{{x_0}^2 + {y_0}^2 - 2x_0}{r^2}$$

satisfy

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2). \qquad \square$$

Generalizations of the idea of the Descartes circle theorem

See “Beyond the Descartes circle theorem” by Jeffrey Lagarias, Colin Mallows, and Allan Wilks

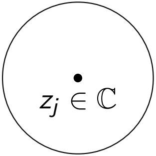
- Published in *The American Mathematical Monthly* in 2002
- Ends in the poem “The Complex Kiss Precise”

Generalization to bend-centers

Yet more is true: if all four discs
Are sited in the complex plane,
Then centers over radii
Obey the self-same rule again.

Figure: An excerpt of "The Complex Kiss Precise" by Lagarias, Mallows, and Wilks.

$b_j = 1/(\text{radius of } j\text{th circle})$



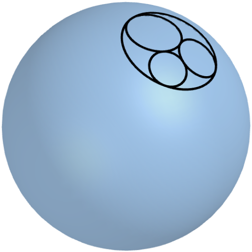
If b_1, b_2, b_3, b_4 and z_1, z_2, z_3, z_4 are bends and centers (respectively) of four mutually tangent circles, then

$$\sum_{j=1}^{4} (b_j z_j)^2 = \frac{1}{2}\left(\sum_{j=1}^{4} b_j z_j\right)^2.$$

Generalization to spherical geometry

Suppose the circles now appear
Upon the surface of a sphere.
Then if by "bend" we mean to say
Cotan of radius, no more,
Then square of sum of "bends" becomes
Two times the sum of squares, plus four.

Figure: An excerpt of "The Complex Kiss Precise" by Lagarias, Mallows, and Wilks.



If r_1, r_2, r_3, r_4 are radii of four mutually tangent circles in spherical geometry and

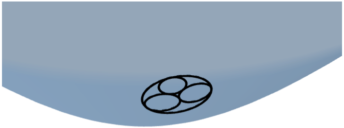
$$b_j = \cot(r_j),$$

then

$$\left(\sum_{j=1}^{4} b_j\right)^2 = 2\left(\sum_{j=1}^{4} {b_j}^2\right) + 4.$$

Now in the hyperbolic plane,
We try to make it work again.
It turns out now by "bend" is meant
The hyperbolic cotangent.
And if we square the sum of those,
Twice sum of squares, less four, it goes.

Figure: An excerpt of "The Complex Kiss Precise" by Lagarias, Mallows, and Wilks.



If r_1, r_2, r_3, r_4 are radii of four mutually tangent circles in hyperbolic geometry and

$$b_j = \coth(r_j),$$

then

$$\left(\sum_{j=1}^{4} b_j\right)^2 = 2\left(\sum_{j=1}^{4} {b_j}^2\right) - 4.$$

Generalization to higher dimensions

And more such wonders can be found
In n dimensions, if allowed.
René Descartes would have been proud.

Figure: An excerpt of "The Complex Kiss Precise" by Lagarias, Mallows, and Wilks.

Generalization to 3-dimensional Euclidean space

To spy out spherical affairs
An oscular surveyor
Might find the task laborious,
The sphere is much the gayer,
And now besides the pair of pairs
A fifth sphere in the kissing shares.
Yet, signs and zero as before,
For each to kiss the other four
The square of the sum of all five bends
Is thrice the sum of their squares.

F. SODDY.

Figure: The last stanza of "The Kiss Precise" by F. Soddy in *Nature*, 1936.

If b_1, b_2, b_3, b_4, b_5 are bends of five mutually tangent spheres, then

$$\left(\sum_{j=1}^{5} b_j\right)^2 = 3\left(\sum_{j=1}^{5} {b_j}^2\right).$$

Generalization to n-dimensional Euclidean space

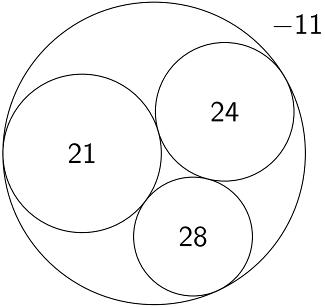
And let us not confine our cares
To simple circles, planes and spheres,
But rise to hyper flats and bends
Where kissing multiple appears.
In n-ic space the kissing pairs
Are hyperspheres, and Truth declares—
As $n + 2$ such osculate
Each with an $n + 1$ fold mate
The square of the sum of all the bends
Is n *times the sum of their squares.*

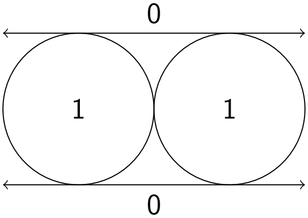
Figure: Thorold Gosset in *Nature*, 1937.

If $b_1, \ldots, b_{n+2}$ are bends of $n+2$ mutually tangent hyperspheres in n-dimensional Euclidean space, then

$$\left(\sum_{j=1}^{n+2} b_j\right)^2 = n\left(\sum_{j=1}^{n+2} {b_j}^2\right).$$

Thank you for listening!





Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four mutually tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Fix b_1, b_2, b_3. What do I know about the solutions to b_4?

Descartes circle theorem

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If b_4 and b_4' are solutions, b_1, b_2, b_3 fixed, then, by the quadratic formula,

$$b_4 + b_4' = 2(b_1 + b_2 + b_3).$$

Matrices and geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$

Matrix form:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}}_{M_4} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

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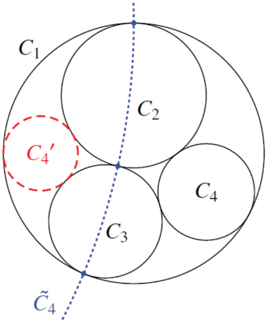

Figure: Four tangent circles and a reflection to a fifth circle.

Matrices and the Apollonian Group

$$M_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & & & \\ 2 & -1 & 2 & 2 \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 2 & 2 & -1 & 2 \\ & & & 1 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

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The **Apollonian group** $\Gamma := \langle M_1, M_2, M_3, M_4 \rangle$ (set of products of M_1, M_2, M_3, M_4)

- maps bends of an Apollonian circle packing to more bends of the packing,
- "generates" all bends of the packing from four bends, and
- sends integer vectors to integer vectors.

Integrality of Bends

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- maps bends of an Apollonian circle packing to more bends of the packing,
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Since we started with an integer vector of bends (namely, $(-11, 21, 24, 28)^t$), all of our bends are integers!

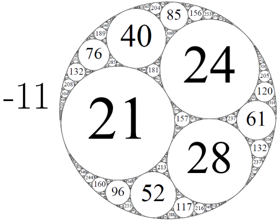

Figure: An integral Apollonian circle packing.