The Kloosterman circle method and weighted representation numbers of positive definite quadratic forms

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Sum of four squares

Which integers can be written (or represented) as the sum of four perfect squares?
That is, which $n \in \mathbb{Z}$ can be written as

$$n = x^2 + y^2 + z^2 + w^2$$

with $x, y, z, w \in \mathbb{Z}$?

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Definition (Representation number for the sum of four squares)

$$r_4(n) = |\{(x, y, z, w)^\top \in \mathbb{Z}^4 : x^2 + y^2 + z^2 + w^2 = n\}|$$
$$= |\{\mathbf{m} \in \mathbb{Z}^4 : f_4(\mathbf{m}) = n\}|,$$

where $f_4(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

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If n is a positive integer, then

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4\nmid d}} d.$$

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What about more general positive definite quadratic forms?

Real quadratic forms

F is a real quadratic form in s variables $\iff$
For all $\mathbf{m} \in \mathbb{R}^s$,

$$F(\mathbf{m}) = \frac{1}{2}\mathbf{m}^\top A\mathbf{m},$$

where A is a real symmetric $s \times s$ matrix and is the Hessian matrix of F.

Example (Example of a quadratic form in 2 variables)

$$\begin{aligned} F(\mathbf{m}) &= m_1^2 + m_1 m_2 + m_2^2 \\ &= \frac{1}{2}\mathbf{m}^\top \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{m} \end{aligned}$$

Quadratic form definitions

Definition (Integral quadratic form)

A quadratic form F is **integral** if $F(\mathbf{m}) \in \mathbb{Z}$ for all $\mathbf{m} \in \mathbb{Z}^s$.

Definition (Positive definite quadratic form)

A quadratic form F is **positive definite** if $F(\mathbf{m}) > 0$ for all $\mathbf{m} \in \mathbb{R}^s \setminus \{\mathbf{0}\}$.

Examples (Examples of positive definite quadratic forms)

- $f_4(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2 + m_4^2$
- $x^2 + xy + y^2$

(Unweighted) representation number

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$$R_F(n) = |\{\mathbf{m} \in \mathbb{Z}^s : F(\mathbf{m}) = n\}|$$

Example

If $F(\mathbf{m}) = f_4(\mathbf{m})$, then $R_F(n) = r_4(n)$.

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$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}},$$

where $\mathbf{1}_{\{F(\mathbf{m})=n\}}$ is the indicator function

$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \begin{cases} 1 & \text{if } F(\mathbf{m}) = n, \\ 0 & \text{otherwise.} \end{cases}$$

Singular series $\mathfrak{S}_F(n)$

The singular series $\mathfrak{S}_F(n)$ contains information about $F(\mathbf{m}) \equiv n \pmod{q}$ for all positive integers q.

There exists a positive integer q such that $F(\mathbf{m}) \equiv n \pmod{q}$ has no solutions
$\implies \mathfrak{S}_F(n) = 0$

An asymptotic for (unweighted) representation numbers

Theorem

Suppose that n is a positive integer.
Suppose that F is a positive definite integral quadratic form in $s \geq 4$ variables.
Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F.
Then the number of integral solutions to $F(\mathbf{m}) = n$ is

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right)$$

for any $\varepsilon > 0$.

- Kloosterman proved this (with a worse error term) in 1926 for diagonal quadratic forms ($F(\mathbf{m}) = a_1 m_1^2 + \cdots + a_s m_s^2$), using what is now called the Kloosterman circle method.
- Obtained as a corollary of my main result.

An asymptotic for (unweighted) representation numbers

- Proofs in
  - §11.4 of *Topics in Classical Automorphic Forms* by Iwaniec
  - §20.4 of *Analytic Number Theory* by Iwaniec and Kowalski

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- Proofs assume equal weight to be given to all integer solutions to $F(\mathbf{m}) = n$

Bump functions & weighted representation numbers

Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}^s$ is denoted by $C_c^\infty(\mathbb{R}^s)$. A function $\psi \in C_c^\infty(\mathbb{R}^s)$ is called a **bump function**.

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Let $\psi \in C_c^\infty(\mathbb{R}^s)$.
For $X > 0$, define

$$\psi_X(\mathbf{m}) = \psi\left(\frac{1}{X}\mathbf{m}\right).$$

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Definition (Weighted representation number)

$$R_{F,\psi,X}(n) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}}\psi_X(\mathbf{m})$$

An asymptotic for weighted representation numbers

Theorem (Heath-Brown, 1996)

Suppose that n is an integer.
Suppose that F is a nonsingular integral quadratic form in $s \geq 4$ variables.
Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$ is a bump function.
Then for $\varepsilon > 0$, the weighted representation number $R_{F,\psi,n^{1/2}}(n)$ is

$$R_{F,\psi,n^{1/2}}(n) = \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n, n^{1/2})n^{\frac{s}{2}-1} + O_{F,\psi,s,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right),$$

where

$$\sigma_{F,\psi,\infty}(n, n^{1/2}) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{|F(\mathbf{m})-1|<\varepsilon} \psi(\mathbf{m})\, d\mathbf{m}.$$

Proof uses the delta method with a Kloosterman refinement.

An asymptotic for weighted representation numbers

For an integer n and $X > 0$, define

$$\sigma_{F,\psi,\infty}(n, X) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{\left|F(\mathbf{m}) - \frac{n}{X^2}\right| < \varepsilon} \psi(\mathbf{m})\, d\mathbf{m}.$$

Example

$$\sigma_{F,\psi,\infty}(n, n^{1/2}) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{|F(\mathbf{m}) - 1| < \varepsilon} \psi(\mathbf{m})\, d\mathbf{m}.$$

An asymptotic for weighted representation numbers

Theorem (J., 2022)

Suppose that n is a positive integer and that F is a positive definite integral quadratic form in $s \geq 4$ variables. Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F. Let λ_s be largest eigenvalue of A. Let L be the smallest positive integer such that $LA^{-1} \in M_s(\mathbb{Z})$. Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$ is a bump function. Then for $X \geq 1/\lambda_s$ and $\varepsilon > 0$, the weighted representation number $R_{F,\psi,X}(n)$ is

$$\begin{aligned} &R_{F,\psi,X}(n) \\ &= \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n,X)X^{s-2} \\ &\quad + O_{\psi,s,\varepsilon}\left(\left(n^{\frac{s}{2}-1} X^{\frac{3-s}{2}+\varepsilon} \lambda_s^{\frac{3-s}{2}+\varepsilon} (\det(A))^{-1/2} + X^{\frac{s-1}{2}+\varepsilon} \lambda_s^{\frac{s+1}{2}+\varepsilon} \right) \right. \\ &\qquad\qquad \left. \times L^{s/2} \tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \right). \end{aligned}$$

An asymptotic for weighted representation numbers

Corollary (J., 2022)

Assume hypotheses of previous theorem and that n is sufficiently large. Set X to be

$$X = n^{1/2}\lambda_s^{(1-s)/(s-2)}(\det(A))^{1/(4-2s)}.$$

Then the weighted representation number $R_{F,\psi,X}(n)$ is

$$R_{F,\psi,X}(n) = \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n,X)X^{s-2} + O_{\psi,s,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\lambda_s^{\frac{s-3-2\varepsilon}{2s-4}}(\det(A))^{\frac{1-s-2\varepsilon}{4s-8}} \times L^{s/2}\prod_{p|2\det(A)}(1-p^{-1/2})^{-1}\right)$$

for any $\varepsilon > 0$.

An asymptotic for a representation number

Corollary

Suppose that n is a positive integer.
Suppose that F is a positive definite integral quadratic form in $s \geq 4$ variables.
Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F.
Then the number of integral solutions to $F(\mathbf{m}) = n$ is

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{s/2-1} + O_{F,\varepsilon}\left(n^{(s-1)/4+\varepsilon}\right)$$

for any $\varepsilon > 0$.

Proof sketch: Choose $X = n^{1/2}$ and ψ to be such that $\psi(\mathbf{m}) = 1$ whenever $\mathbf{m} \in \mathbb{R}^s$ satisfies $F(\mathbf{m}) = 1$.

Proof sketch of main result

1. Write $R_{F,\psi,X}(n)$ as

$$R_{F,\psi,X}(n) = \int_0^1 \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathrm{e}(x(F(\mathbf{m}) - n))\, \psi_X(\mathbf{m})\; dx,$$

where $\mathrm{e}(z) = e^{2\pi i z}$.

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5. Use bounds on oscillatory integrals to bound the archimedean parts. (The principle of nonstationary phase is used.)
6. Put estimates together and compute the main term.

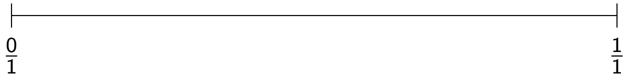
Thank you for listening!

Farey sequence $\mathfrak{F}_Q$ of order Q

Definition

For $Q \geq 1$, the **Farey sequence** $\mathfrak{F}_Q$ **of order** Q is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 1$



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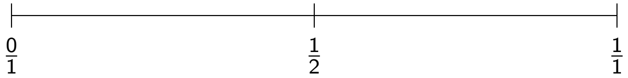

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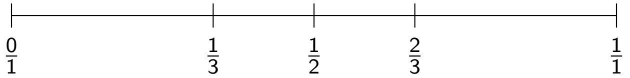
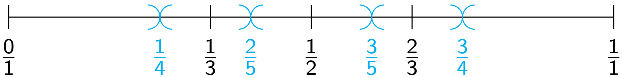
$Q = 3$



An example of a Farey dissection



Lemma for Kloosterman circle method

Lemma

Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic function of period 1 and with real Fourier coefficients (so that $\overline{f(x)} = f(-x)$ for all $x \in \mathbb{R}$). Then

$$\int_0^1 f(x)\, dx = 2\,\mathrm{Re}\left(\sum_{1 \le q \le Q} \int_0^{\frac{1}{qQ}} \sum_{\substack{Q < d \le q+Q \\ qdx < 1 \\ \gcd(d,q)=1}} f\left(x - \frac{d^*}{q}\right)\, dx \right),$$

where d^* is the multiplicative inverse of d modulo q.

Use this for

$$f(x) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathrm{e}(x(F(\mathbf{m}) - n))\, \psi_X(\mathbf{m}).$$

Arithmetic and archimedean parts

$$R_{F,\psi,X}(n) = 2\,\mathrm{Re}\left(\sum_{1\le q\le Q}\frac{1}{q^s}\int_0^{\frac{1}{qQ}} \mathrm{e}(-nx)\sum_{\mathbf{r}\in\mathbb{Z}^s}\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)\,T_{\mathbf{r}}(q,n;x)\;dx\right),$$

where the arithmetic part is

$$T_{\mathbf{r}}(q,n;x) = \sum_{\substack{Q<d\le q+Q\\ qdx<1\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) G_{\mathbf{r}}(-d^*,q),$$

the Gauss sum $G_{\mathbf{r}}(d,q)$ is

$$G_{\mathbf{r}}(d,q) = \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{h})+\mathbf{h}\cdot\mathbf{r})\right),$$

and the archmedean part is

$$\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q) = \int_{\mathbb{R}^s}\mathrm{e}\left(xF(\mathbf{m})-\frac{1}{q}\mathbf{m}\cdot\mathbf{r}\right)\psi_X(\mathbf{m})\;d\mathbf{m}.$$

A potential application: A strong asymptotic local-global principle for certain Kleinian sphere packings

Examples of Kleinian sphere packings that have or might have a strong asymptotic local-global principle:

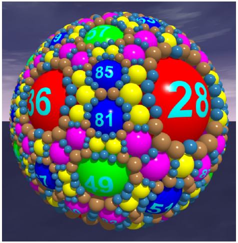


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

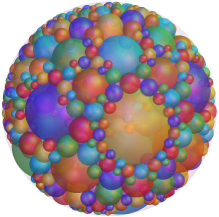


Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

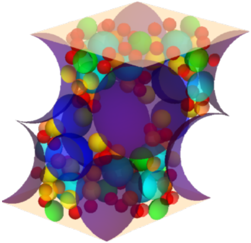
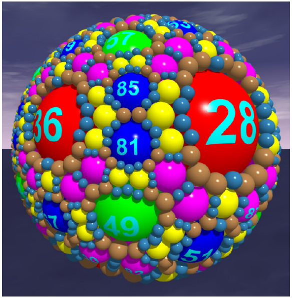


Figure: A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.

Soddy sphere packings



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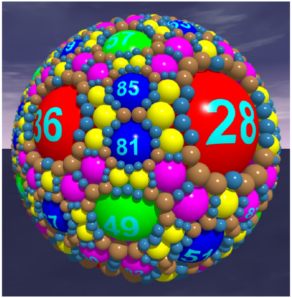
Label on sphere:

bend $= 1/\text{radius}$

All of the bends of this Soddy sphere packing are integers.

Soddy sphere packings



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Which integers appear as bends?

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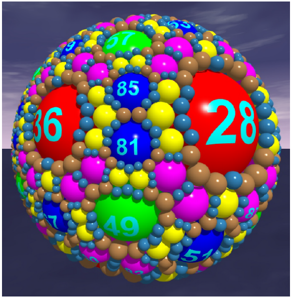

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Label on sphere:
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All of the bends of this Soddy sphere packing are integers.

Which integers appear as bends?

Are there any congruence or local obstructions?

Admissible integers

Definition (Admissible integers)

Let $\mathcal{P}$ be an integral Kleinian sphere packing in $\mathbb{R}^d \cup \{\infty\}$.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some } (d-1)\text{-sphere in } \mathcal{P} \pmod{q}.$$

Equivalently, m is admissible if m has no local obstructions.

Admissible integers

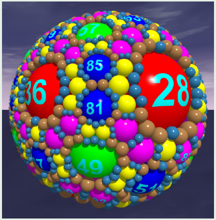
Theorem (Kontorovich, 2019)

m is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod{3},$$

where $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ depends only on the packing.

Example



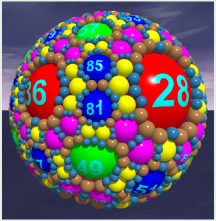
m is admissible $\iff$
$m \equiv 0$ or $1 \pmod{3}$.

A strong asymptotic local-global theorem

Theorem (Kontorovich, 2019)

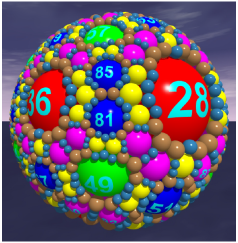
The bends of a fixed primitive integral Soddy sphere packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a sphere in the packing.

Example



If $m \equiv 0$ or $1 \pmod 3$ and m is sufficiently large, then m is the bend of a sphere in the packing.

Examples of integral Kleinian sphere packings



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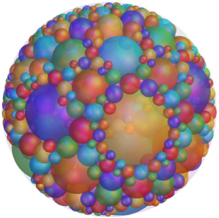

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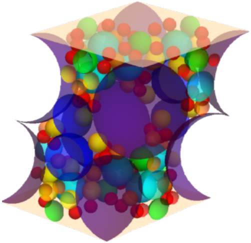

Figure: A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.

Goal: Prove strong asymptotic local-global principles for certain integral Kleinian sphere packings, that is, prove:
If m is admissible and sufficiently large, then m is the bend of an $(d-1)$-sphere in the packing.

Definition (Admissible integers)

Let $\mathcal{P}$ be an integral Kleinian sphere packing in $\mathbb{R}^d \cup \{\infty\}$.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some } (d-1)\text{-sphere in } \mathcal{P} \pmod{q}.$$

A strong asymptotic local-global conjecture

Conjecture (A strong asymptotic local-global conjecture for certain Kleinian sphere packings)

Let $\mathcal{P}$ be a primitive integral Kleinian $(d-1)$-sphere packing in $\mathbb{R}^d \cup \{\infty\}$ with an orientation-preserving automorphism group Γ of Möbius transformations.

Then every sufficiently large admissible integer is a bend of a $(d-1)$-sphere in $\mathcal{P}$. That is, there exists an $N_0 = N_0(\mathcal{P})$ such that if m is admissible and $m > N_0$, then m is the bend of a $(d-1)$-sphere in $\mathcal{P}$.

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1. *Suppose that there exists a $(d-1)$-sphere $S_0 \in \mathcal{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$, where K is an imaginary quadratic field and $\mathcal{O}_K$ is the ring of integers of K. This condition implies that $d \geq 3$.*

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2. *Suppose that there is a $(d-1)$-sphere $S_1 \in \mathcal{P}$ that is tangent to S_0.*

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- Potentially, my version of the Kloosterman circle method could be then used to prove a result towards a strong asymptotic local-global conjecture for certain Kleinian sphere packings.
- The potential result would be the first to apply to multiple conformally inequivalent integral Kleinian sphere packings.

The singular series and the real factor

Singular series:

$$\mathfrak{S}_F(n) = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q} \left(F(\mathbf{h}) - n \right) \right)$$

Real factor:

$$\sigma_{F,\psi,\infty}(n, X) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{\left| F(\mathbf{m}) - \frac{n}{X^2} \right| < \varepsilon} \psi(\mathbf{m}) \, d\mathbf{m}.$$

$$\kappa_{s,q}(a,b) = \sum_{d \pmod q} \left(\frac{d}{q}\right)^{s} \mathrm{e}\left(\frac{ad+bd^*}{q}\right) \tag{1}$$

is either a Kloosterman sum (if s is even) or a Salié sum (if s is odd).

Lemma (Weil bound for Kloosterman sums)

If s is even, a and b are integers, and q is a positive integer, then

$$|\kappa_{s,q}(a,b)| \leq \tau(q)(\gcd(a,b,q))^{1/2}q^{1/2},$$

where the divisor function $\tau(q)$ is the number of positive divisors of q.

A principle of nonstationary phase

Theorem (Principle of nonstationary phase in 1 variable, J., 2022)

Let $\psi \in C_c^\infty(\mathbb{R})$ *and let* $M \geq 0$. *Let* $f \in C^\infty(\mathbb{R})$ *be such that* $|f'(x)| \geq B > 0$ *and* $|f^{(j)}(x)| \leq |f'(x)|$ *for all* $x \in \operatorname{supp}(\psi)$ *and for each integer* j *satisfying* $2 \leq j \leq \lceil M \rceil$. *Then*

$$\int_{\mathbb{R}} \mathrm{e}(f(x))\, \psi(x)\; dx \ll_{\psi, M} B^{-M}.$$

Kleinian sphere packings

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An $(d-1)$-sphere packing $\mathcal{P}$ is **Kleinian** if its limit set is that of a geometrically finite group $\Gamma < \mathrm{Isom}(\mathbb{H}^{d+1})$.

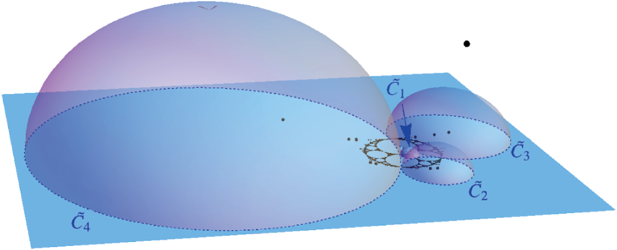

Figure: Apollonian circle packing as the limit set of Γ. Image by Alex Kontorovich.

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- Γ is a thin group.