The Kloosterman circle method and weighted representation numbers of positive definite quadratic forms

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February 27, 2023

Sum of four squares

Which integers can be written (or represented) as the sum of four perfect squares?
That is, which $n \in \mathbb{Z}$ can be written as

$$n = x^2 + y^2 + z^2 + w^2$$

with $x, y, z, w \in \mathbb{Z}$?

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Every nonnegative integer can be written as the sum of four perfect squares.

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Examples (Examples of integers written as the sum of four squares)

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$$\begin{aligned} 7 &= 2^2 + 1^2 + 1^2 + 1^2 \\ &= 1^2 + 2^2 + 1^2 + 1^2. \end{aligned}$$

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Definition (Representation number for the sum of four squares)

$$r_4(n) = |\{(x, y, z, w)^\top \in \mathbb{Z}^4 : x^2 + y^2 + z^2 + w^2 = n\}|$$
$$= |\{\mathbf{m} \in \mathbb{Z}^4 : f_4(\mathbf{m}) = n\}|,$$

where $f_4(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

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$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d.$$

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What about more general positive definite quadratic forms?

Real quadratic forms

F is a real quadratic form in s variables $\iff$
For all $\mathbf{m} \in \mathbb{R}^s$,

$$F(\mathbf{m}) = \frac{1}{2}\mathbf{m}^\top A\mathbf{m},$$

where A is a real symmetric $s \times s$ matrix and is the Hessian matrix of F.

Example (Example of a quadratic form in 2 variables)

$$\begin{aligned} F(\mathbf{m}) &= m_1^2 + m_1 m_2 + m_2^2 \\ &= \frac{1}{2}\mathbf{m}^\top \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{m} \end{aligned}$$

Quadratic form definitions

Definition (Integral quadratic form)

A quadratic form F is **integral** if $F(\mathbf{m}) \in \mathbb{Z}$ for all $\mathbf{m} \in \mathbb{Z}^s$.

Definition (Positive definite quadratic form)

A quadratic form F is **positive definite** if $F(\mathbf{m}) > 0$ for all $\mathbf{m} \in \mathbb{R}^s \setminus \{\mathbf{0}\}$.

Examples (Examples of integral positive definite quadratic forms)

- $f_4(\mathbf{m}) = m_1^2 + m_2^2 + m_3^2 + m_4^2$
- $x^2 + xy + y^2$

(Unweighted) representation number

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Example

If $F(\mathbf{m}) = f_4(\mathbf{m})$, then $R_F(n) = r_4(n)$.

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$$R_F(n) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}},$$

where $\mathbf{1}_{\{F(\mathbf{m})=n\}}$ is the indicator function

$$\mathbf{1}_{\{F(\mathbf{m})=n\}} = \begin{cases} 1 & \text{if } F(\mathbf{m}) = n, \\ 0 & \text{otherwise.} \end{cases}$$

Singular series $\mathfrak{S}_F(n)$

The singular series $\mathfrak{S}_F(n)$ contains information about $F(\mathbf{m}) \equiv n \pmod{q}$ for all positive integers q.

$\mathfrak{S}_F(n) = 0 \iff$
there exists a positive integer q such that $F(\mathbf{m}) \equiv n \pmod{q}$ has no solutions

Big O notation

$f(x) = O(g(x))$ means that there exists a constant $C > 0$ such that

$$|f(x)| \leq Cg(x)$$

for all $x \in D$, where D is an appropriate domain that can be deduced from the context.

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- If $x \geq 1$, then $x = O(x^2)$ since $|x| \leq x^2$ for $x \geq 1$.
- If $x \geq 1$, then $x^2 + x = O(x^2)$ since $|x^2 + x| \leq 2x^2$ for $x \geq 1$.
- If $0 < \varepsilon < 1$, then $\varepsilon^2 = O(\varepsilon)$ since $|\varepsilon^2| \leq \varepsilon$ for $0 < \varepsilon < 1$.

An asymptotic for (unweighted) representation numbers

Theorem

Suppose that n is a positive integer.
Suppose that F is a positive definite integral quadratic form in $s \geq 4$ variables.
Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F.
Then the number of integral solutions to $F(\mathbf{m}) = n$ is

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{\frac{s}{2}-1} + O_{F,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right)$$

for any $\varepsilon > 0$.

- Kloosterman proved this (with a worse error term) in 1926 for diagonal quadratic forms ($F(\mathbf{m}) = a_1 m_1^2 + \cdots + a_s m_s^2$), using what is now called the Kloosterman circle method.
- Obtained as a corollary of my main result.

An asymptotic for (unweighted) representation numbers

- Proofs in
  - §11.4 of *Topics in Classical Automorphic Forms* by Iwaniec
  - §20.4 of *Analytic Number Theory* by Iwaniec and Kowalski

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- Proofs use the Kloosterman circle method
- Proofs assume equal weight to be given to all integer solutions to $F(\mathbf{m}) = n$

Bump functions & weighted representation numbers

Definition (Bump function)

The space of real-valued, infinitely differentiable, and compactly supported functions on $\mathbb{R}^s$ is denoted by $C_c^\infty(\mathbb{R}^s)$. A function $\psi \in C_c^\infty(\mathbb{R}^s)$ is called a **bump function**.

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Let $\psi \in C_c^\infty(\mathbb{R}^s)$.
For $X > 0$, define

$$\psi_X(\mathbf{m}) = \psi\left(\frac{1}{X}\mathbf{m}\right).$$

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Definition (Weighted representation number)

$$R_{F,\psi,X}(n) = \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathbf{1}_{\{F(\mathbf{m})=n\}}\psi_X(\mathbf{m})$$

An asymptotic for weighted representation numbers

Theorem (Heath-Brown, 1996)

Suppose that n is an integer.
Suppose that F is a nonsingular integral quadratic form in $s \geq 4$ variables.
Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$ is a bump function.
Then for $\varepsilon > 0$, the weighted representation number $R_{F,\psi,n^{1/2}}(n)$ is

$$R_{F,\psi,n^{1/2}}(n) = \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n, n^{1/2})n^{\frac{s}{2}-1} + O_{F,\psi,s,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\right),$$

where

$$\sigma_{F,\psi,\infty}(n, X) = \lim_{\varepsilon\to 0^+} \frac{1}{2\varepsilon} \int_{\left|F(\mathbf{m})-\frac{n}{X^2}\right|<\varepsilon} \psi(\mathbf{m})\, d\mathbf{m}.$$

Proof uses the delta method with a Kloosterman refinement.

An asymptotic for weighted representation numbers

Theorem (J., 2022)

Suppose that n is a positive integer and that F is a positive definite integral quadratic form in $s \geq 4$ variables. Let $A \in M_s(\mathbb{Z})$ be the Hessian matrix of F. Let λ_s be largest eigenvalue of A. Let L be the smallest positive integer such that $LA^{-1} \in M_s(\mathbb{Z})$. Suppose that $\psi \in C_c^\infty(\mathbb{R}^s)$ is a bump function. Then for $X \geq 1/\lambda_s$ and $\varepsilon > 0$, the weighted representation number $R_{F,\psi,X}(n)$ is

$$\begin{aligned}
&R_{F,\psi,X}(n) \\
&= \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n,X)X^{s-2} \\
&\quad + O_{\psi,s,\varepsilon}\Bigg(\left(n^{\frac{s}{2}-1} X^{\frac{3-s}{2}+\varepsilon} \lambda_s^{\frac{3-s}{2}+\varepsilon} (\det(A))^{-1/2} + X^{\frac{s-1}{2}+\varepsilon} \lambda_s^{\frac{s+1}{2}+\varepsilon} \right) \\
&\qquad\qquad \times L^{s/2}\tau(n) \prod_{p \mid 2\det(A)} (1-p^{-1/2})^{-1} \Bigg).
\end{aligned}$$

An asymptotic for weighted representation numbers

Corollary (J., 2022)

Assume hypotheses of previous theorem and that n is sufficiently large. Set X to be

$$X = n^{1/2}\lambda_s^{(1-s)/(s-2)}(\det(A))^{1/(4-2s)}.$$

Then the weighted representation number $R_{F,\psi,X}(n)$ is

$$R_{F,\psi,X}(n) = \mathfrak{S}_F(n)\sigma_{F,\psi,\infty}(n,X)X^{s-2} + O_{\psi,s,\varepsilon}\left(n^{\frac{s-1}{4}+\varepsilon}\lambda_s^{\frac{s-3-2\varepsilon}{2s-4}}(\det(A))^{\frac{1-s-2\varepsilon}{4s-8}} \times L^{s/2}\prod_{p\mid 2\det(A)}(1-p^{-1/2})^{-1}\right)$$

for any $\varepsilon > 0$.

Corollary

Suppose that n is a positive integer.
Suppose that F is a positive definite integral quadratic form in $s \geq 4$ variables.
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Then the number of integral solutions to $F(\mathbf{m}) = n$ is

$$R_F(n) = \mathfrak{S}_F(n) \frac{(2\pi)^{s/2}}{\Gamma(s/2)\sqrt{\det(A)}} n^{s/2-1} + O_{F,\varepsilon}\left(n^{(s-1)/4+\varepsilon}\right)$$

for any $\varepsilon > 0$.

Proof sketch: Choose $X = n^{1/2}$ and ψ to be such that $\psi(\mathbf{m}) = 1$ whenever $\mathbf{m} \in \mathbb{R}^s$ satisfies $F(\mathbf{m}) = 1$.

Proof sketch of main result

1. Write $R_{F,\psi,X}(n)$ as

$$R_{F,\psi,X}(n) = \int_0^1 \sum_{\mathbf{m}\in\mathbb{Z}^s} \mathrm{e}(x(F(\mathbf{m}) - n))\,\psi_X(\mathbf{m})\ dx,$$

where $\mathrm{e}(z) = e^{2\pi i z}$.

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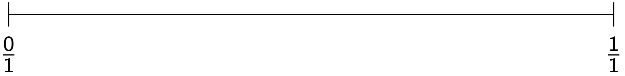
5. Use bounds on oscillatory integrals to bound the archimedean parts. (The principle of nonstationary phase is used.)
6. Put estimates together and compute the main term.

Farey sequence $\mathfrak{F}_Q$ of order Q

Definition

For $Q \geq 1$, the **Farey sequence** $\mathfrak{F}_Q$ **of order** Q is the increasing sequence of all reduced fractions $\frac{a}{q}$ with $1 \leq q \leq Q$ and $\gcd(a, q) = 1$.

$Q = 1$



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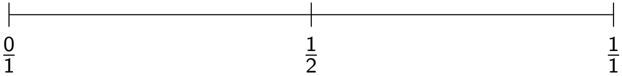

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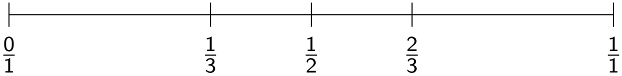

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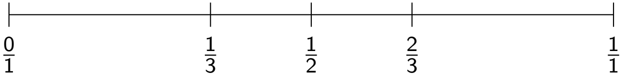
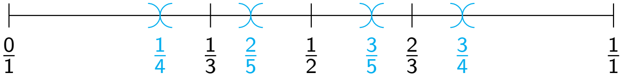


Example of Farey dissection when $Q = 3$:



Lemma

Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic function of period 1 and with real Fourier coefficients (so that $\overline{f(x)} = f(-x)$ for all $x \in \mathbb{R}$). Then

$$\int_0^1 f(x)\,dx = 2\,\mathrm{Re}\left(\sum_{1 \le q \le Q} \int_0^{\frac{1}{qQ}} \sum_{\substack{Q < d \le q+Q \\ qdx < 1 \\ \gcd(d,q)=1}} f\left(x - \frac{d^*}{q}\right)\,dx \right),$$

where d^* is the multiplicative inverse of d modulo q.

Use this for

$$f(x) = \sum_{\mathbf{m} \in \mathbb{Z}^s} \mathrm{e}(x(F(\mathbf{m}) - n))\,\psi_X(\mathbf{m}).$$

Arithmetic and archimedean parts

$$R_{F,\psi,X}(n) = 2\,\mathrm{Re}\left(\sum_{1\le q\le Q} \frac{1}{q^s} \int_0^{\frac{1}{qQ}} \mathrm{e}(-nx) \sum_{\mathbf{r}\in\mathbb{Z}^s} \mathcal{I}_{F,\psi}(x,X,\mathbf{r},q)\, T_{\mathbf{r}}(q,n;x)\ dx \right),$$

where the arithmetic part is

$$T_{\mathbf{r}}(q,n;x) = \sum_{\substack{Q<d\le q+Q\\ qdx<1\\ \gcd(d,q)=1}} \mathrm{e}\left(n\frac{d^*}{q}\right) G_{\mathbf{r}}(-d^*,q),$$

the Gauss sum $G_{\mathbf{r}}(d,q)$ is

$$G_{\mathbf{r}}(d,q) = \sum_{\mathbf{h}\in(\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{1}{q}(dF(\mathbf{h}) + \mathbf{h}\cdot\mathbf{r})\right),$$

and the archmedean part is

$$\mathcal{I}_{F,\psi}(x,X,\mathbf{r},q) = \int_{\mathbb{R}^s} \mathrm{e}\left(xF(\mathbf{m}) - \frac{1}{q}\mathbf{m}\cdot\mathbf{r}\right) \psi_X(\mathbf{m})\ d\mathbf{m}.$$

A potential application: A strong asymptotic local-global principle for certain Kleinian sphere packings

Examples of Kleinian sphere packings that have or might have a strong asymptotic local-global principle:

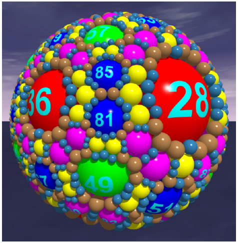


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

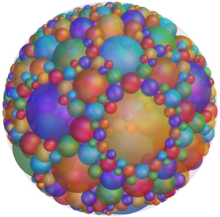


Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

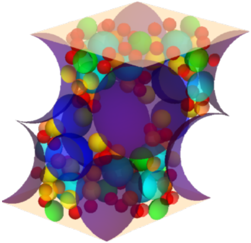
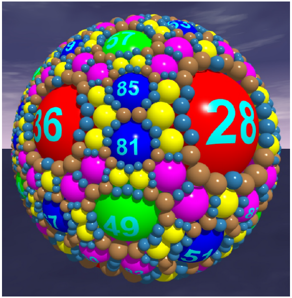


Figure: A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.

Soddy sphere packings



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Label on sphere:
bend = 1/radius

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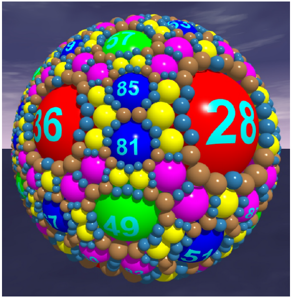

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All of the bends of this Soddy sphere packing are integers.

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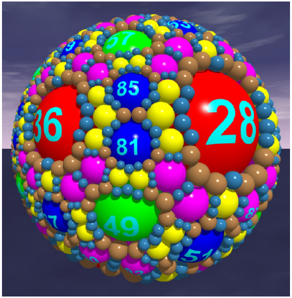

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Which integers appear as bends?

Soddy sphere packings

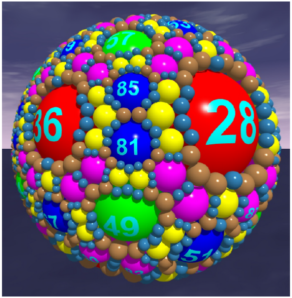


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

Label on sphere:
bend = 1/radius

All of the bends of this Soddy sphere packing are integers.

Which integers appear as bends?

Are there any congruence or local obstructions?

Admissible integers

Definition (Admissible integers)

Let $\mathcal{P}$ be an integral Kleinian sphere packing in $\mathbb{R}^d \cup \{\infty\}$.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some } (d-1)\text{-sphere in } \mathcal{P} \pmod{q}.$$

Equivalently, m is admissible if m has no local obstructions.

Admissible integers

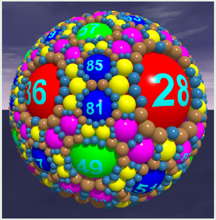
Theorem (Kontorovich, 2019)

m is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod 3,$$

where $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ depends only on the packing.

Example



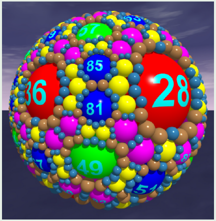
m is admissible $\iff$ $m \equiv 0$ or $1 \pmod 3$.

A strong asymptotic local-global theorem

Theorem (Kontorovich, 2019)

The bends of a fixed primitive integral Soddy sphere packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a sphere in the packing.

Example



If $m \equiv 0$ or $1 \pmod{3}$ and m is sufficiently large, then m is the bend of a sphere in the packing.

Examples of integral Kleinian sphere packings

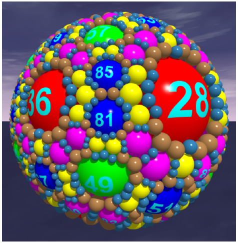


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

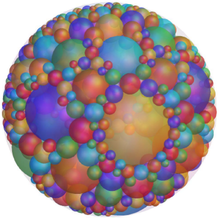


Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

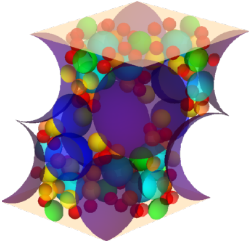


Figure: A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.

Strong asymptotic local-global principles

Goal: Prove strong asymptotic local-global principles for certain integral Kleinian sphere packings, that is, prove:
If m is admissible and sufficiently large, then m is the bend of an $(d-1)$-sphere in the packing.

Definition (Admissible integers)

Let $\mathcal{P}$ be an integral Kleinian sphere packing in $\mathbb{R}^d \cup \{\infty\}$.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some } (d-1)\text{-sphere in } \mathcal{P} \pmod{q}.$$

A strong asymptotic local-global conjecture

Conjecture (A strong asymptotic local-global conjecture for certain Kleinian sphere packings)

Let $\mathcal{P}$ be a primitive integral Kleinian $(d-1)$-sphere packing in $\mathbb{R}^d \cup \{\infty\}$ with an orientation-preserving automorphism group Γ of Möbius transformations.
Under some conditions, every sufficiently large admissible integer is a bend of a $(d-1)$-sphere in $\mathcal{P}$. That is, there exists an $N_0 = N_0(\mathcal{P})$ such that if m is admissible and $m > N_0$, then m is the bend of a $(d-1)$-sphere in $\mathcal{P}$.

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- Potentially, my version of the Kloosterman circle method could be then used to prove a result towards a strong asymptotic local-global conjecture for certain Kleinian sphere packings.
- The potential result would be the first to apply to multiple conformally inequivalent integral Kleinian sphere packings.

Thank you for listening!

The singular series and the real factor

Singular series:

$$\mathfrak{S}_F(n) = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{d \in (\mathbb{Z}/q\mathbb{Z})^\times} \sum_{\mathbf{h} \in (\mathbb{Z}/q\mathbb{Z})^s} \mathrm{e}\left(\frac{d}{q} \left(F(\mathbf{h}) - n \right) \right)$$

Real factor:

$$\sigma_{F,\psi,\infty}(n, X) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{\left| F(\mathbf{m}) - \frac{n}{X^2} \right| < \varepsilon} \psi(\mathbf{m}) \, d\mathbf{m}.$$

Kloosterman sums and Salié sums

$$\kappa_{s,q}(a,b) = \sum_{d \pmod q} \left(\frac{d}{q}\right)^s \mathrm{e}\left(\frac{ad+bd^*}{q}\right) \tag{1}$$

is either a Kloosterman sum (if s is even) or a Salié sum (if s is odd).

Lemma (Weil bound for Kloosterman sums)

If s is even, a and b are integers, and q is a positive integer, then

$$|\kappa_{s,q}(a,b)| \leq \tau(q)(\gcd(a,b,q))^{1/2} q^{1/2},$$

where the divisor function $\tau(q)$ is the number of positive divisors of q.

A principle of nonstationary phase

Theorem (Principle of nonstationary phase in 1 variable, J., 2022)

Let $\psi \in C_c^\infty(\mathbb{R})$ *and let* $M \geq 0$. *Let* $f \in C^\infty(\mathbb{R})$ *be such that* $|f'(x)| \geq B > 0$ *and* $|f^{(j)}(x)| \leq |f'(x)|$ *for all* $x \in \operatorname{supp}(\psi)$ *and for each integer* j *satisfying* $2 \leq j \leq \lceil M \rceil$. *Then*

$$\int_{\mathbb{R}} \mathrm{e}(f(x))\, \psi(x)\, dx \ll_{\psi,M} B^{-M}.$$

Kleinian sphere packings

Definition (Kleinian sphere packing)

An $(d-1)$-sphere packing $\mathcal{P}$ is **Kleinian** if its limit set is that of a geometrically finite group $\Gamma < \mathrm{Isom}(\mathbb{H}^{d+1})$.

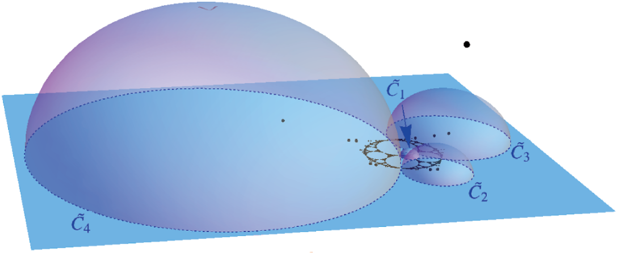

Figure: Apollonian circle packing as the limit set of Γ. Image by Alex Kontorovich.

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- Γ is a thin group.

A strong asymptotic local-global conjecture

Conjecture (A strong asymptotic local-global conjecture for certain Kleinian sphere packings)

Let $\mathcal{P}$ be a primitive integral Kleinian $(d-1)$-sphere packing in $\mathbb{R}^d \cup \{\infty\}$ with an orientation-preserving automorphism group Γ of Möbius transformations.

Then every sufficiently large admissible integer is a bend of a $(d-1)$-sphere in $\mathcal{P}$. That is, there exists an $N_0 = N_0(\mathcal{P})$ such that if m is admissible and $m > N_0$, then m is the bend of a $(d-1)$-sphere in $\mathcal{P}$.

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1. *Suppose that there exists a $(d-1)$-sphere $S_0 \in \mathcal{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$, where K is an imaginary quadratic field and $\mathcal{O}_K$ is the ring of integers of K. This condition implies that $d \geq 3$.*

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2. *Suppose that there is a $(d-1)$-sphere $S_1 \in \mathcal{P}$ that is tangent to S_0.*

Then every sufficiently large admissible integer is a bend of a $(d-1)$-sphere in $\mathcal{P}$. That is, there exists an $N_0 = N_0(\mathcal{P})$ such that if m is admissible and $m > N_0$, then m is the bend of a $(d-1)$-sphere in $\mathcal{P}$.

Soddy sphere packings: The construction

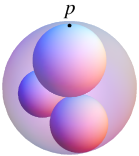

Figure: Four mutually tangent spheres.

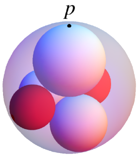

Figure: Four tangent spheres with two additional spheres.

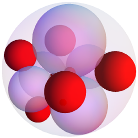

Figure: More spheres.

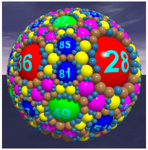

Figure: A Soddy sphere packing.