Local-global I: Apollonian packings

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Thin Groups and Super-approximation
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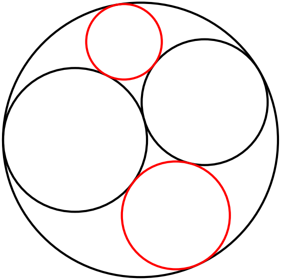
Apollonian circle packings: The construction

Given three mutually tangent circles with disjoint points of tangency, there are exactly two circles tangent to the given ones. (Proved by Apollonius of Perga.)



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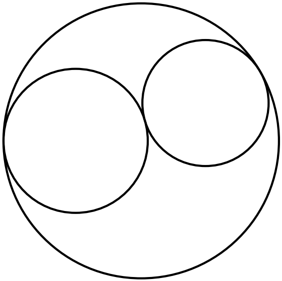

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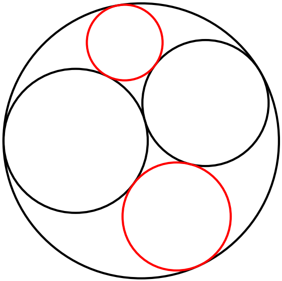

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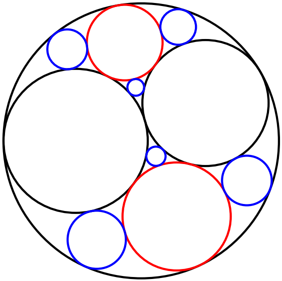

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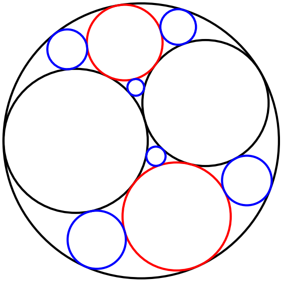


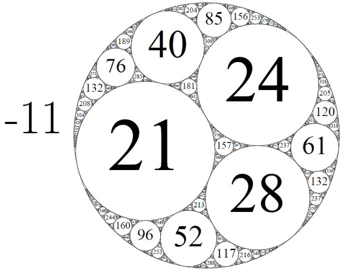
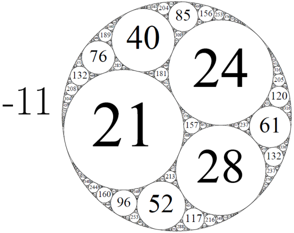


Figure: An Apollonian circle packing.

Apollonian circle packings



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Label on circle:

$$\text{bend} = \text{curvature}$$
$$= 1/\text{radius}$$

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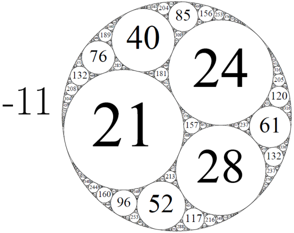

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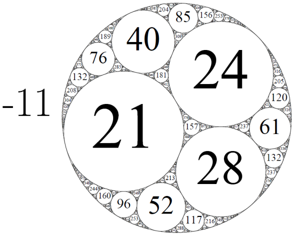

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Why?

“The Kiss Precise” by F. Soddy

Four circles to the kissing come.
The smaller are the benter.
The bend is just the inverse of
The distance from the centre.
Though their intrigue left Euclid dumb
There’s now no need for rule of thumb.

Since zero bend’s a dead straight line
And concave bends have minus sign,
The sum of the squares of all four bends
Is half the square of their sum.

Figure: An excerpt of “The Kiss Precise” by F. Soddy in *Nature*, 1936.

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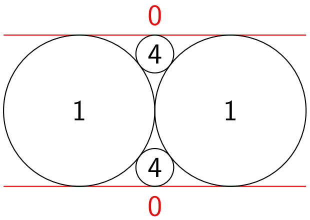

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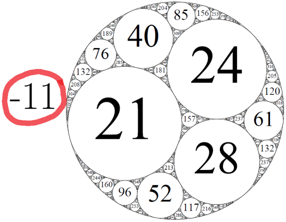

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If b_1, b_2, b_3, b_4 are bends of four mutually tangent circles, then

$$b_1^2 + b_2^2 + b_3^2 + b_4^2 = \frac{1}{2}(b_1 + b_2 + b_3 + b_4)^2.$$

Descartes circle theorem

Theorem (Descartes circle theorem, 1643)

If b_1, b_2, b_3, b_4 are bends of four mutually tangent circles, then

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

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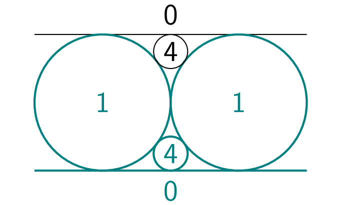

$b_1 = 0,\ b_2 = b_3 = 1,\ b_4 = 4$

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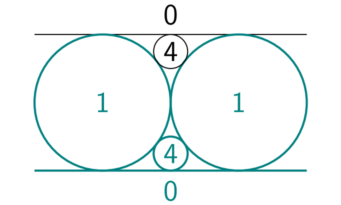

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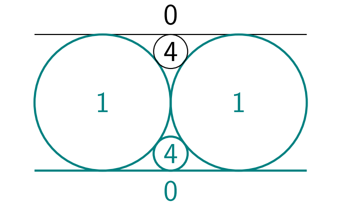

$b_1 = 0,\ b_2 = b_3 = 1,\ b_4 = 4$

$(0 + 1 + 1 + 4)^2 = 6^2 = 36$

$2(0^2 + 1^2 + 1^2 + 4^2) = 2(18) = 36$

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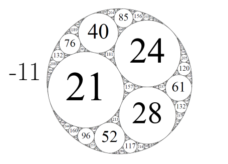

$$b_1 = -11, \; b_2 = 21, \; b_3 = 24, \; b_4 = 28$$

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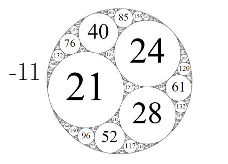

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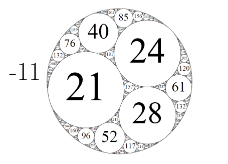

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Fix b_1, b_2, b_3. What do I know about the solutions to b_4?

If b_4 and b_4' are solutions, b_1, b_2, b_3 fixed, then, by the quadratic formula,

$$b_4 + b_4' = 2(b_1 + b_2 + b_3).$$

Matrices and geometry

$$b_4' = 2b_1 + 2b_2 + 2b_3 - b_4$$

Matrix form:

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}}_{M_4} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

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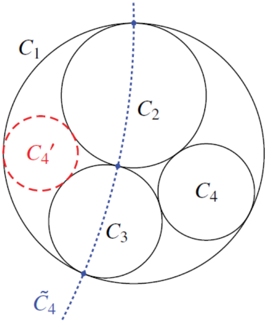

Figure: Four tangent circles and a reflection to a fifth circle.

Matrices and the Apollonian group

$$M_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & & & \\ 2 & -1 & 2 & 2 \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 2 & 2 & -1 & 2 \\ & & & 1 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

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The **Apollonian group** $\Gamma := \langle M_1, M_2, M_3, M_4 \rangle$

- maps bends of an Apollonian circle packing to more bends of the packing,
- "generates" all bends of the packing from four bends, and
- sends integer vectors to integer vectors.

Integrality of bends

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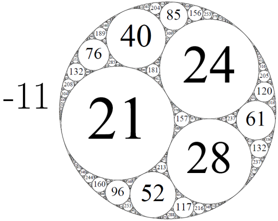
Since we start with an integer vector of bends (namely, $(-11, 21, 24, 28)^\top$), all of our bends are integers!



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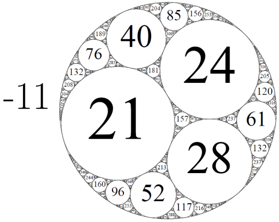

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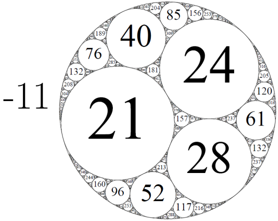


Figure: An Apollonian circle packing.

Which integers appear as bends?

Are there any congruence or local obstructions?

Admissible integers

Definition (Admissible integers for Apollonian circle packings)

Let $\mathcal{P}$ be an integral Apollonian circle packing.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some circle in } \mathcal{P} \pmod{q}.$$

Equivalently, m is admissible if m has no local obstructions.

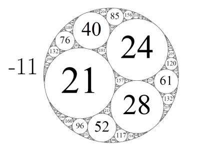
Admissible integers

Theorem (Fuchs, 2011)

Let $\mathcal{P}$ be a primitive integral Apollonian circle packing. Then m is admissible if and only if m is in certain congruence classes modulo 24.

(The congruence classes depend on the packing.)

Example



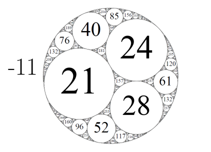
m is admissible $\iff$
$m \equiv 0, 4, 12, 13, 16,$ or $21 \pmod{24}$.

Local-global conjecture

Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, 2003)

The bends of a fixed primitive integral Apollonian circle packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a circle in the packing.

Example



We think that if
$m \equiv 0, 4, 12, 13, 16,$ or $21 \pmod{24}$
and m is sufficiently large,
then m is the bend of a circle in the packing.
We do not have a proof of this!

Why do we have a local-global conjecture?

Theorem (Kontorovich–Oh, 2011)

The number of circles in an Apollonian circle packing $\mathcal{P}$ with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^{δ}, where $\delta =$ the Hausdorff dimension of the closure of $\mathcal{P}$.

For Apollonian circle packings, we have

$$\delta \approx 1.30568\ldots$$

Thus, we would would expect that the multiplicity of a given admissible bend up to N is roughly $N^{\delta-1} \approx N^{0.30568} \geq 1$, so we should expect that every sufficiently large admissible number to be represented.

First observation

Observation (Graham–Lagarias–Mallows–Wilks–Yan, 2003)

At least $c_1 N^{1/2}$ of all integers less than N appear as bends in a fixed primitive integral Apollonian circle packing.

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Proof comes from looking at the largest entries of $(M_1 M_2)^k \mathbf{v}_0$, where $\mathbf{v}_0$ is the root quadruple of bends (e.g., $(-11, 21, 24, 28)^\top$) and $k > 0$.

These largest entries grow like k^2.

Powers of $M_4 M_3$

$$(M_4 M_3)^k = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 4k^2 - 2k & 4k^2 - 2k & 1 - 2k & 2k \\ 4k^2 + 2k & 4k^2 + 2k & -2k & 2k + 1 \end{pmatrix}$$

so that

$$\mathbf{e}_4^\top (M_4 M_3)^k \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = 4(b_1 + b_2)k^2 + 2(b_1 + b_2 - b_3 + b_4)k + b_4$$

is a bend in our packing, where $\mathbf{e}_4 = (0, 0, 0, 1)^\top$.

Sarnak's letter

Theorem (Sarnak, 2007)

At least $c_2 N/\sqrt{\log(N)}$ of all integers less than N appear as bends in a fixed primitive integral Apollonian circle packing.

Spin homomorphisms

Descartes quadratic form (with signature (3,1)):

$$Q(\mathbf{v}) = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2) - (b_1 + b_2 + b_3 + b_4)^2$$

There is spin homomorphism $\rho\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_Q(\mathbb{R})$ such that

$$\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mapsto M_4 M_3 \qquad \text{and}$$

$$\pm\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mapsto M_2 M_3.$$

(Uses spin homomorphisms from $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}(3,1)$ and from $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}(2,1)$.)

A congruence subgroup

$\pm\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$ and $\pm\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}$ generate

$$\Lambda(2) = \left\{\begin{pmatrix}a & b\\c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix}a & b\\c & d\end{pmatrix} \equiv \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \pmod 2\right\}.$$

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$\implies \rho(\Lambda(2)) < \Gamma$ and $\rho(\Lambda(2))$ fixes C_1.

A congruence subgroup

For any $x, y \in \mathbb{Z}$ and $\gcd(2x, y) = 1$, there is a matrix of the form $\begin{pmatrix} * & 2x \\ * & y \end{pmatrix} \in \Lambda(2)$.

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$\implies$

$$\mathbf{e}_4^\top \rho\left(\begin{pmatrix} * & 2x \\ * & y \end{pmatrix}\right) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = 4(b_1 + b_2)x^2 + 2(b_1 + b_2 - b_3 + b_4)xy + (b_1 + b_4)y^2 - b_1 \quad (*)$$

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Number of integers up to N represented by $(*)$ with $\gcd(2x, y) = 1$ is of order $N/\sqrt{\log(N)}$.

Positive density result

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Obtained by looking at multiple orbits of $\rho(\Lambda(2))$ in the packing.

The best we can do right now

Theorem (Bourgain–Kontorovich, 2014)

Almost every admissible number is the bend of a circle in the Apollonian circle packing $\mathcal{P}$. Quantitatively, the number of exceptions up to N is bounded by $O(N^{1-\eta})$, where $\eta > 0$ is effectively computable.

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Proof uses the circle method with other tools, including spectral theory and expander graphs.

Proof ideas

Representation number:

$$R_N(m) = \sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} \sum_{\substack{g \in \rho(\Lambda(2)) \\ \|g\| < X}} \mathbf{1}_{\{m = \mathbf{e}_4^\top \gamma g \mathbf{v}_0\}}$$

where m is of size N, T and X depend on N, $\mathbf{v}_0$ is a root quadruple, and

$$\mathbf{1}_{\{m = \mathbf{e}_4^\top \gamma g \mathbf{v}_0\}} = \begin{cases} 1 & \text{if } m = \mathbf{e}_4^\top \gamma g \mathbf{v}_0, \\ 0 & \text{otherwise.} \end{cases}$$
$$= \int_0^1 e^{2\pi i \theta (\mathbf{e}_4^\top \gamma g \mathbf{v}_0 - m)} \, d\theta.$$

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Want to know when $R_N(m) > 0$

Proof ideas

$$S_N(\theta) = \sum_{m \in \mathbb{Z}} R_N(m) \mathrm{e}(m\theta) = \sum_{\substack{\gamma \in \Gamma \\ \|\gamma\| < T}} \sum_{\substack{g \in \rho(\Lambda(2)) \\ \|g\| < X}} \mathrm{e}\left(\theta \mathbf{e}_4^\top \gamma g \mathbf{v}_0\right)$$

where $\mathrm{e}(z) = e^{2\pi i z}$.

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Representation number:

$$\begin{aligned} R_N(m) &= \int_0^1 S_N(\theta)\mathrm{e}(-m\theta)\ d\theta \\ &= \mathcal{M}_N(m) + \mathcal{E}_N(m) \end{aligned}$$

where

- $\mathcal{M}_N(m)$: "main" term
- $\mathcal{E}_N(m)$: "error" term

Proof ideas

$$R_N(m) = \mathcal{M}_N(m) + \mathcal{E}_N(m)$$

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  - $\mathcal{M}_N(m) = \int_{\mathfrak{M}} S_N(\theta)\mathrm{e}(-m\theta)\ d\theta$
  - $\mathfrak{M}$: major arcs, intervals close to rationals with small denominators
- $\mathcal{E}_N(m)$: "error" term

Proof ideas

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    - $\mathcal{M}_N(m) \gg \mathfrak{S}(m)N^{\delta-1}$
    - $\mathfrak{S}(m)$: singular series, responsible for admissibility conditions
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  - $\displaystyle\sum_{m\in\mathbb{Z}} |\mathcal{E}_N(m)|^2 = \int_{\mathfrak{m}} |S_N(\theta)|^2\ d\theta = o\left(N^{2\delta-1}\right)$

Proof ideas

(not exactly what was done but the main ideas)

$$\mathcal{M}_N(m) \gg \mathfrak{S}(m) N^{\delta-1}$$

$$\sum_{m\in\mathbb{Z}} |\mathcal{E}_N(m)|^2 = o\left(N^{2\delta-1}\right)$$

$E(N) = \{ m \le N : m$ is admissible but not represented $\}$

WTS: $\#E(N) = o(N)$

$m \in E(N) \Rightarrow \mathcal{E}_N(m) \gg N^{\delta-1}$

$\Rightarrow 1 \ll \dfrac{|\mathcal{E}_N(m)|^2}{N^{2\delta-2}}$

$\Rightarrow \#E(N) \ll N^{2-2\delta} \sum_{m\in\mathbb{Z}} |\mathcal{E}_N(m)|^2 = o(N)$

Proof ideas

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- Sum over γ
- Sum over g

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  - Captures admissibility conditions in major arcs
  - Uses expander graphs and the spectral gap
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$$R_N(m) = \int_0^1 S_N(\theta)\mathrm{e}(-m\theta)\ d\theta$$

- Sum over γ
  - Captures admissibility conditions in major arcs
  - Uses expander graphs and the spectral gap
- Sum over g
  - Provides sufficient cancellation in minor arcs
  - Uses shifted binary quadratic forms

Kleinian sphere packings

Definition (Kleinian sphere packing)

An $(n-1)$-sphere packing $\mathcal{P}$ is **Kleinian** if its limit set is that of a geometrically finite group $\Gamma < \mathrm{Isom}(\mathcal{H}^{n+1})$.

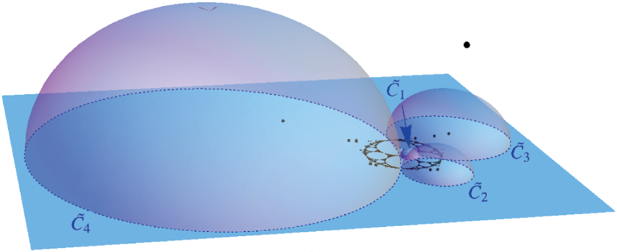

Figure: Apollonian circle packing as the limit set of Γ. Figure created by Alex Kontorovich.

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- Γ is a thin group.

Result for Kleinian 1-Sphere (Circle) Packings

Theorem (Fuchs–Stange–Zhang, 2019)

If $\mathcal{P}$ is a primitive integral Kleinian 1-sphere packing in $\widehat{\mathbb{R}^2}$ satisfying certain conditions, almost every admissible number is the bend of a circle in $\mathcal{P}$.

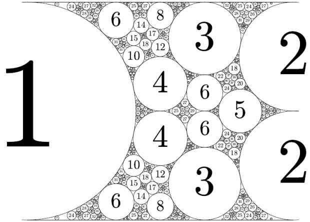


Figure: An integral Kleinian (more specifically, cuboctahedral) 1-sphere packing that satisfies the conditions of the theorem. Figure taken from "Local-Global Principles in Circle Packings" by Fuchs, Stange, and Zhang.

What happens in higher dimensions?

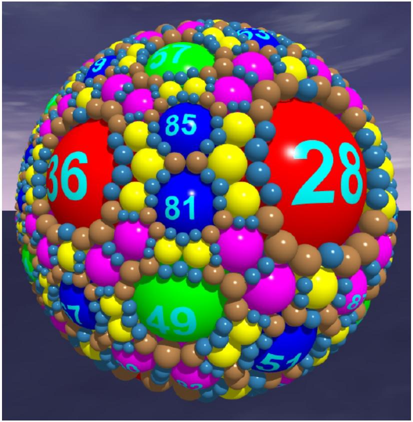


Figure: A Soddy sphere packing made by Nicolas Hannachi.

Soddy sphere packings: The construction

Given four mutually tangent spheres with disjoint points of tangency, there are exactly two spheres tangent to the given ones.

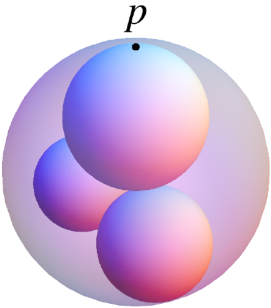


Figure: Four tangent spheres.

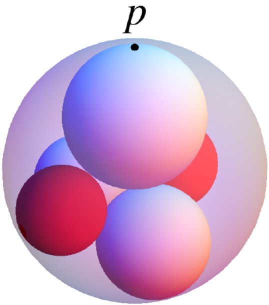
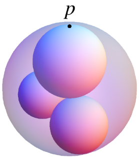


Figure: Four tangent spheres with two additional tangent spheres.



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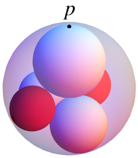

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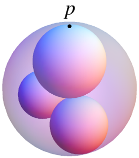

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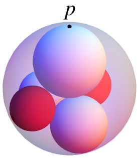


Figure: Four tangent spheres with two additional tangent spheres.

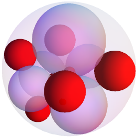


Figure: More tangent spheres.

Soddy sphere packings: The construction

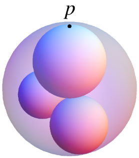


Figure: Four tangent spheres.

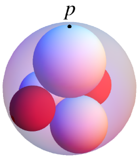
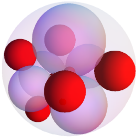


Figure: Four tangent spheres with two additional tangent spheres.



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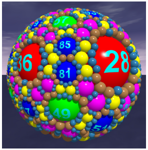


Figure: A Soddy sphere packing.

Soddy sphere packings: The construction

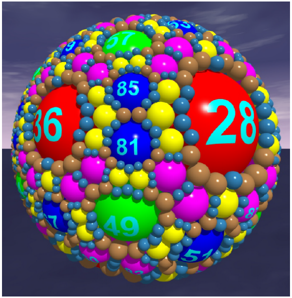
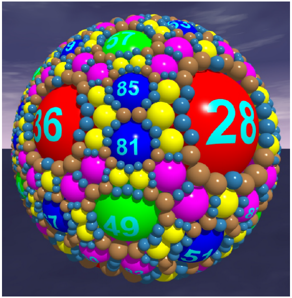


Figure: An integral Soddy sphere packing made by Nicolas Hannachi.

Label on sphere:

bend $= 1/\text{radius}$



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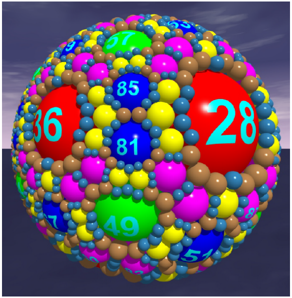
All of the bends of this Soddy sphere packing are integers.



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Which integers appear as bends?

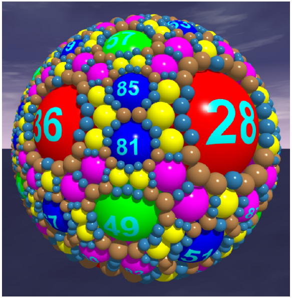

Figure: An integral Soddy sphere packing made by Nicolas Hannachi.

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Which integers appear as bends?

Are there any congruence or local obstructions?

Admissible integers

Definition (Admissible integers for Soddy sphere packings)

Let $\mathcal{P}$ be an integral Soddy sphere packing.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some sphere in } \mathcal{P} \pmod{q}.$$

Equivalently, m is admissible if m has no local obstructions.

Admissible integers

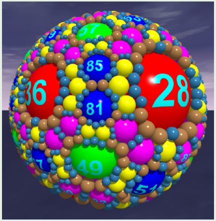
Theorem (Kontorovich, 2019)

m is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod{3},$$

where $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ depends only on the packing.

Example



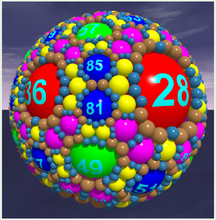
m is admissible $\iff$
$m \equiv 0$ or $1 \pmod{3}$.

A local-global theorem

Theorem (Kontorovich, 2019)

The bends of a fixed primitive integral Soddy sphere packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a sphere in the packing.

Example



If $m \equiv 0$ or $1 \pmod 3$ and m is sufficiently large, then m is the bend of a sphere in the packing.

My research

Goal: Extend Kontorovich's result and prove a a strong asymptotic local-global principle for bends of certain integral Kleinian sphere packings in dimension at least 3.

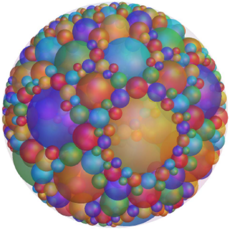


Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing made by Kei Nakamura.

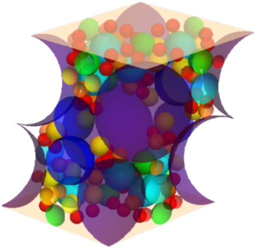


Figure: A fundamental domain of an integral Kleinian sphere packing made by Arseniy (Senia) Sheydvasser.

Illustrations Credits

Besides the illustrations previously credited and a few Apollonian circle packing construction illustrations created by the presenter, the illustrations for this talk came from the following papers:

- Jean Bourgain and Alex Kontorovich, "On the local-global conjecture for integral Apollonian gaskets," *Inventiones mathematicae*, volume 196, pp. 589–650, 2014.
- Alex Kontorovich, "From Apollonius to Zaremba: Local-global phenomena in thin orbits," *Bulletin of the American Mathematical Society*, volume 50, number 2, pp. 187-228, 2013, `https://www.ams.org/journals/bull/2013-50-02/S0273-0979-2013-01402-2/`.
- Alex Kontorovich, "The Local-Global Principle for Integral Soddy Sphere Packings," *Journal of Modern Dynamics*, volume 15, pp. 209-236, 2019, `https://www.aimsciences.org/article/doi/10.3934/jmd.2019019`.

Thank you for listening!