The Local-Global Principle for Integral Crystallographic Sphere Packings

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AMS Special Session on Quadratic Forms and Theta Functions
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Given four mutually tangent spheres with disjoint points of tangency, there are exactly two spheres tangent to the given ones.

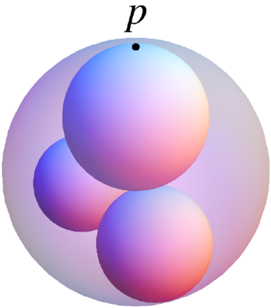


Figure: Four tangent spheres.

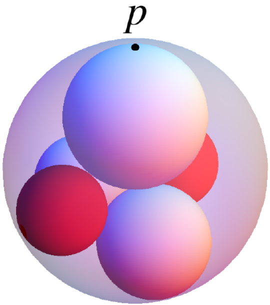
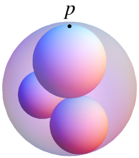


Figure: Four tangent spheres with two additional tangent spheres.

Soddy Sphere Packings: The Construction



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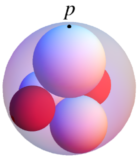

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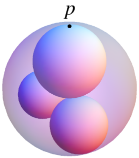

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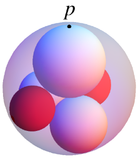

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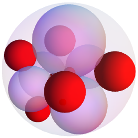

Figure: More tangent spheres.

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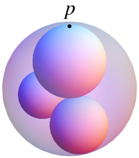

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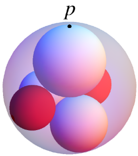

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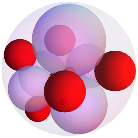

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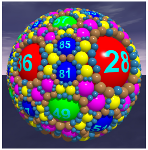

Figure: A Soddy sphere packing.

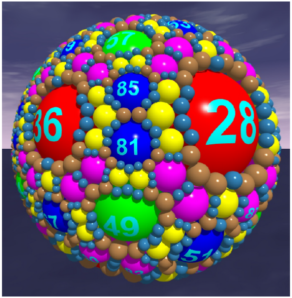

Figure: A Soddy sphere packing made by Nicolas Hannachi.

Label on sphere:
bend $= 1/\text{radius}$

What do you notice about the bends that you can see on this Soddy sphere packing?

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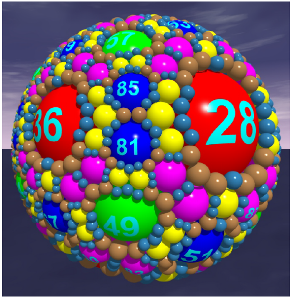

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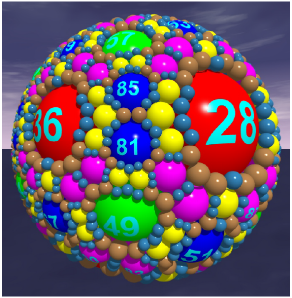

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What do you notice about the bends that you can see on this Soddy sphere packing?

They are all integers.

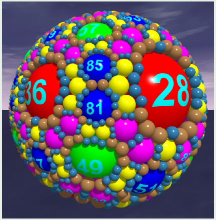
Which integers appear as bends?

Lemma (Kontorovich, 2019)

For a primitive integral Soddy sphere packing $\mathscr{P}$, there is an $\varepsilon = \varepsilon(\mathscr{P}) \in \{\pm 1\}$ such that each bend of the packing is

$$\equiv 0 \text{ or } \varepsilon \pmod{3}.$$

Example



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Definition

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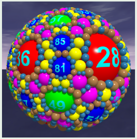


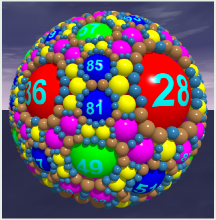
m is admissible $\iff$
$m \equiv 0$ or $1 \pmod{3}$.

Local-Global Theorem

Theorem (Kontorovich, 2019)

The bends of a fixed primitive integral Soddy sphere packing $\mathscr{P}$ *satisfy a local-to-global principle.*
That is, there is an $N_0 = N_0(\mathscr{P})$ *so that, if* $m > N_0$ *and* m *is admissible, then* m *is the bend of a sphere in the packing.*

Example



If $m \equiv 0$ or $1 \pmod 3$ and m is sufficiently large, then m is the bend of a sphere in the packing.

My Research

Goal: Prove a local-global principle for bends of more general integral sphere packings (called crystallographic sphere packings).

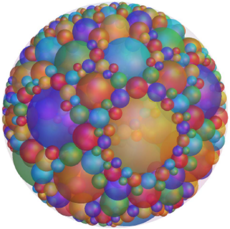


Figure: An integral crystallographic (more specifically, an orthoplicial) packing made by Kei Nakamura.

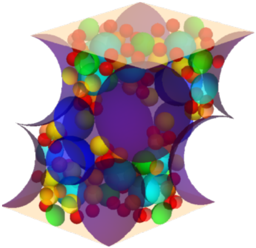


Figure: A fundamental domain of an integral crystallographic packing made by Arseniy (Senia) Sheydvasser.

Möbius Transformations

Automorphism group Γ of Möbius transformations that map a packing $\mathscr{P}$ to itself

$$\begin{aligned} \Gamma : \widehat{\mathbb{R}^n} &\to \widehat{\mathbb{R}^n} \\ z &\mapsto g(z) = (a \cdot z + b)(c \cdot z + d)^{-1}, \\ g &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \end{aligned}$$

a, b, c, d in a Clifford algebra (which is the set of quaternions when $n = 3$)

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- Stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$, K imaginary quadratic field.

Then every sufficiently large admissible integer is a bend of a $(n-1)$-sphere in $\mathscr{P}$.

Congruence Subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$

Definition

A *principal congruence subgroup* of $\mathrm{PSL}_2(\mathcal{O}_K)$ is a subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ of the form

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varrho} \right\}$$

for a fixed $\varrho \in \mathcal{O}_K$.

Example (Soddy sphere packing, Kontorovich, 2019)

There exists a sphere $S_0 \in \mathscr{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) : b, c \equiv 0 \pmod{\varrho} \right\},$$

where $\mathcal{O} = \mathbb{Z}[e^{\pi i/3}]$ and $\varrho = 1 + e^{\pi i/3}$.

Proof Methods Outline

1. The assumption that Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ shows that the set of bends of $\mathscr{P}$ contains the "primitive" values of a shifted quaternary quadratic form.

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2. This shifted quaternary quadratic form gives you enough to work with so that you can apply the circle method to show that every sufficiently large admissible number is represented as a bend.
   - Major arcs: use spectral theory
   - Minor arcs: use Kloosterman circle method

Quadratic Form for Soddy Sphere Packings

Example (Quadratic form for Soddy sphere packings)

Shifted quaternary quadratic form in $a_0, a_1, c_0, c_1 \in \mathbb{Z}$:

$$\hat{\beta}|C(a_0 + a_1\omega) + D\varrho(c_0 + c_1\omega)|^2 - Dj\bar{C} + Cj\bar{D},$$

$\omega = e^{\pi i/3}$
$\varrho = 1 + \omega$
$\gcd(a_0 + a_1\omega, \varrho(c_0 + c_1\omega)) = 1$

$\hat{\beta} \in \mathbb{R}$ and C, D in the Clifford algebra depend on packing.

(Scale appropriately to obtain a primitive integral quadratic form.)

Illustrations Credit

Alex Kontorovich, "The Local-Global Principle for Integral Soddy Sphere Packings," *Journal of Modern Dynamics*, 2019, `https://www.aimsciences.org/article/doi/10.3934/jmd.2019019`.

Many of (but not all of) the pictures used in this presentation are from this paper.

Thank you for listening!

Crystallographic Sphere Packing

Definition

A $(n-1)$-sphere packing is *crystallographic* if its limit set is that of a geometrically finite reflection group $\Gamma < \text{Isom}(\mathbb{H}^{n+1})$.

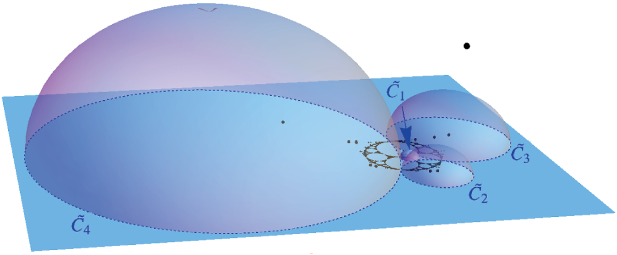

Figure: Apollonian circle packing as the limit set of Γ. Figure created by Alex Kontorovich.