An Asymptotic Local-Global Principle for Integral Kleinian Sphere Packings

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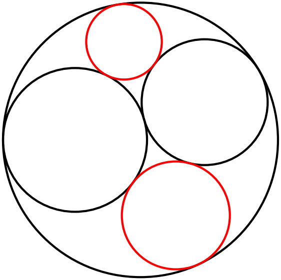
Mid-Atlantic Seminar On Numbers V
March 28, 2021

Apollonian Circle Packings: The Construction

Given three mutually tangent circles with disjoint points of tangency, there are exactly two circles tangent to the given ones. (Proved by Apollonius of Perga.)



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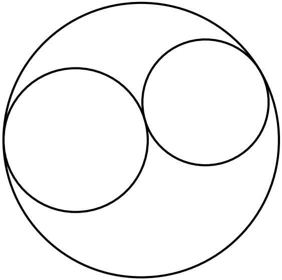

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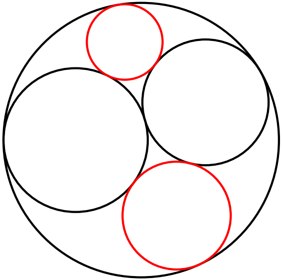

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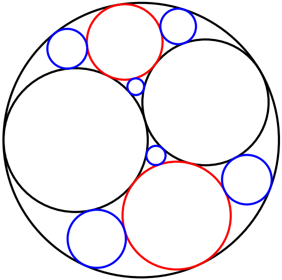

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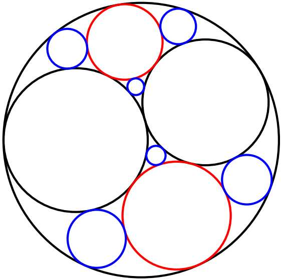


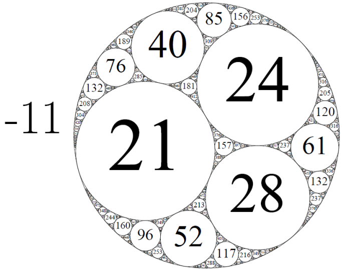


Figure: An Apollonian circle packing.

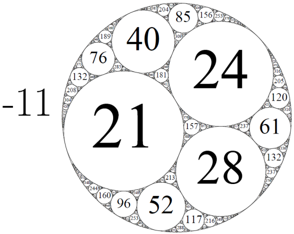
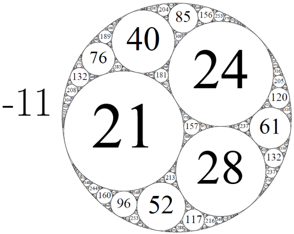


Figure: An Apollonian circle packing.

Label on circle:

bend = 1/radius



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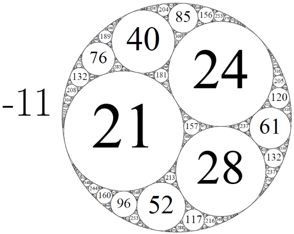

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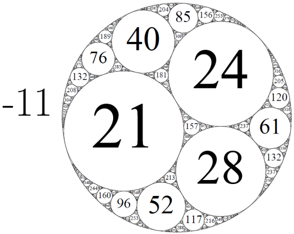
Which integers appear as bends?



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Which integers appear as bends?

Are there any congruence or local obstructions?

Admissible Integers

Definition (Admissible integers for Apollonian circle packings)

Let $\mathcal{P}$ be an integral Apollonian circle packing.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some circle in } \mathcal{P} \pmod{q}.$$

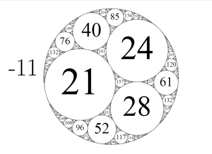
Equivalently, m is admissible if m has no local obstructions.

Admissible Integers

Theorem (Fuchs, 2011)

m is admissible if and only if m is in certain congruence classes modulo 24.
(The congruence classes depend on the packing.)

Example



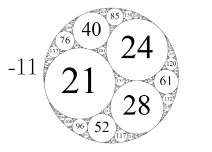
m is admissible $\iff$
$m \equiv 0, 4, 12, 13, 16, \text{ or } 21 \pmod{24}$.

Local-Global Conjecture

Conjecture (Graham–Lagarias–Mallows–Wilks–Yan, 2003)

The bends of a fixed primitive integral Apollonian circle packing $\mathcal{P}$ satisfy an asymptotic local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a circle in the packing.

Example



We think that if
$m \equiv 0, 4, 12, 13, 16, \text{ or } 21 \pmod{24}$
and m is sufficiently large,
then m is the bend of a circle in the packing.
We do not have a proof of this!

Why Do We Have a Local-Global Conjecture?

Theorem (Kontorovich–Oh, 2011)

The number of circles in an Apollonian circle packing $\mathcal{P}$ with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^{δ}, where $\delta =$ the Hausdorff dimension of the closure of $\mathcal{P}$.

For Apollonian circle packings, we have

$$\delta \approx 1.30568\ldots$$

Thus, we would would expect that the multiplicity of a given admissible bend up to N is roughly $N^{\delta-1} \approx N^{0.30568} \geq 1$, so we should expect that every sufficiently large admissible number to be represented.

The Best We Can Do Right Now

Theorem (Bourgain–Kontorovich, 2014)

Almost every admissible number is the bend of a circle in a fixed primitive integral Apollonian circle packing $\mathcal{P}$. Quantitatively, the number of exceptions up to N is bounded by $O(N^{1-\eta})$, where $\eta > 0$ is effectively computable.

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Proof outline:

1. Show that the automorphism group of the Apollonian circle packing contains the congruence subgroup $\Gamma(2)$ of $PSL_2(\mathbb{Z})$, and $\Gamma(2)$ is the stabilizer of a particular circle. This implies that the set of bends contains primitive values of a shifted binary quadratic form. (Sarnak, 2007)

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2. The shifted binary quadratic form gives you enough to work with so that you can apply the circle method (with some other tools, including spectral theory, used in the major arc and minor arc analyses) to obtain an "almost all" statement.

Given four mutually tangent spheres with disjoint points of tangency, there are exactly two spheres tangent to the given ones.

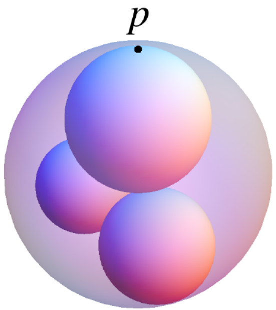


Figure: Four tangent spheres.

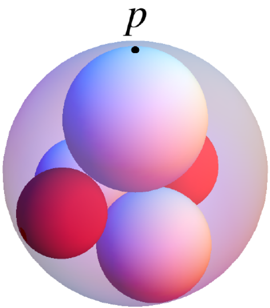
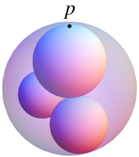


Figure: Four tangent spheres with two additional tangent spheres.

Soddy Sphere Packings: The Construction



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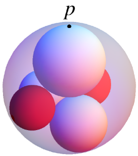

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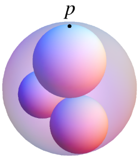

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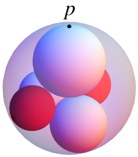

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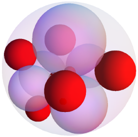

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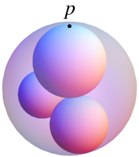

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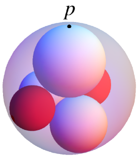

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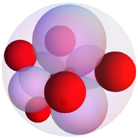

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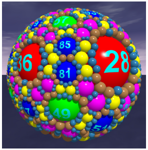

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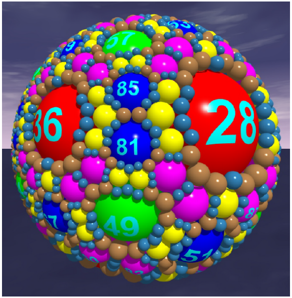

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Label on sphere:
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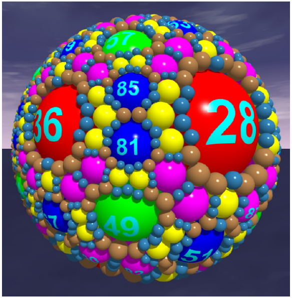

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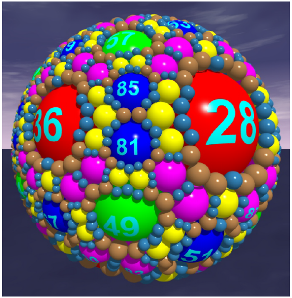

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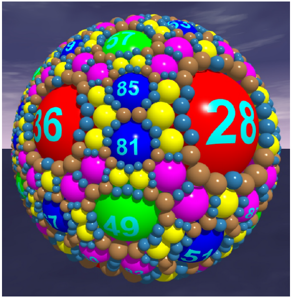

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Equivalently, m is admissible if m has no local obstructions.

Admissible Integers

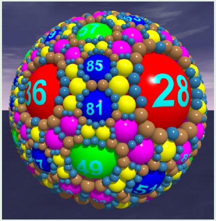
Theorem (Kontorovich, 2019)

m is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod 3,$$

where $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ depends only on the packing.

Example



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An Asymptotic Local-Global Theorem

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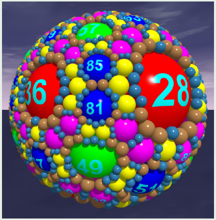

If $m \equiv 0$ or $1 \pmod 3$ and m is sufficiently large, then m is the bend of a sphere in the packing.

Proof Outline for Soddy Sphere Packing Result

1. Show that the automorphism group of the Soddy sphere packing contains a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains "primitive" values of a shifted quaternary quadratic form.

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2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the circle method to show that every sufficiently large admissible number is represented by the quadratic form without the primitivity restriction.

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2. The shifted quaternary quadratic form gives you enough to work with so that you can quote the circle method to show that every sufficiently large admissible number is represented by the quadratic form without the primitivity restriction.
3. Show that the singular series (with the primitivity restriction) is bounded away from zero.

Congruence Subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$

Definition (Principal congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$)

For an imaginary quadratic field K, a **principal congruence subgroup** of $\mathrm{PSL}_2(\mathcal{O}_K)$ is a subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ of the form

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varrho} \right\}$$

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Example (Soddy sphere packing, Kontorovich, 2019)

There exists a sphere $S_0 \in \mathcal{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) : b, c \equiv 0 \pmod{\varrho} \right\},$$

where $\mathcal{O} = \mathbb{Z}[e^{\pi i/3}]$ and $\varrho = (1 + e^{\pi i/3})$.

Definition (Kleinian sphere packing)

A $(n-1)$-sphere packing $\mathcal{P}$ is **Kleinian** if its limit set is that of a geometrically finite group $\Gamma < \text{Isom}(\mathcal{H}^{n+1})$.

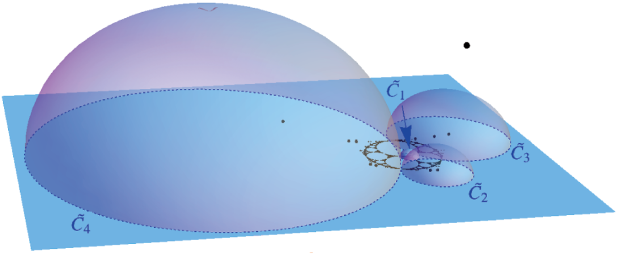

Figure: Apollonian circle packing as the limit set of Γ.

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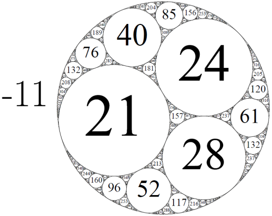
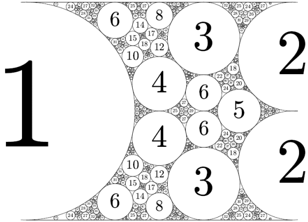
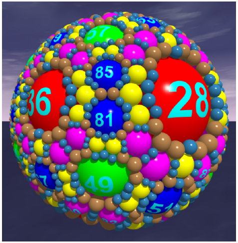
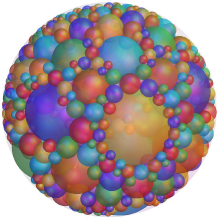
- Action of $\mathrm{Isom}(\mathcal{H}^{n+1})$ extends continuously to $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, the boundary of $\mathcal{H}^{n+1}$.
- Γ stabilizes $\mathcal{P}$ (i.e., Γ maps $\mathcal{P}$ to itself).

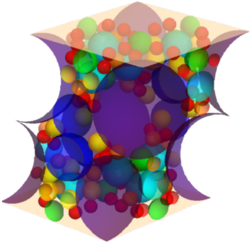
Examples of Kleinian Sphere Packings











We want to prove asymptotic local-global principles for certain integral Kleinian sphere packings, that is, we want to prove the following:
If m is admissible and sufficiently large, then m is a bend of the packing.

Definition (Admissible integers)

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An integer m is **admissible (or locally represented)** if for every $q \geq 1$

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Group of Möbius Transformations

Group of orientation-preserving Möbius transformations $\text{Möb}(\mathcal{H}^{n+1}) < \text{Isom}(\mathcal{H}^{n+1})$

$$\begin{aligned} \text{Möb}(\mathcal{H}^{n+1}) : \mathcal{H}^{n+1} &\to \mathcal{H}^{n+1} \\ z &\mapsto g(z) = (a \cdot z + b)(c \cdot z + d)^{-1}, \\ g &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Möb}(\mathcal{H}^{n+1}) \end{aligned}$$

a, b, c, d in a Clifford algebra C_{n-1} with some restrictions.
Action extends to $\widehat{\mathbb{R}^n}$.

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Clifford algebras for small n:

- $C_1 = \mathbb{C}$
- $C_2 = \mathbb{H}$, the set of quaternions

Group of Möbius Transformations

Example ($n = 2$)

$\text{Möb}(\mathcal{H}^3) \cong \text{PSL}_2(\mathbb{C})$ acts on $\widehat{\mathbb{R}^2} \cong \widehat{\mathbb{C}}$ via

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Restrictions make defining what $\text{Möb}(\mathcal{H}^{n+1})$ is isomorphic to trickier for $n > 2$.
(For example, $\text{Möb}(\mathcal{H}^4) \ncong \text{PSL}_2(\mathbb{H})$, even though $C_2 = \mathbb{H}$.)

Result for Kleinian 1-Sphere (Circle) Packings

Theorem (Fuchs–Stange–Zhang, 2019)

Almost every admissible number is the bend of a circle in $\mathcal{P}$. Quantitatively, the number of exceptions up to N is bounded by $O(N^{1-\eta})$.

Generalizes Bourgain–Kontorovich result for Apollonian circle packings to other Kleinian 1-sphere packings.

Result for Kleinian 1-Sphere (Circle) Packings

Theorem (Fuchs–Stange–Zhang, 2019)

Suppose that

- $\mathcal{P}$ *is a primitive integral Kleinian* 1*-sphere packing in* $\widehat{\mathbb{R}^2}$ *with an automorphism group* $\Gamma < \mathrm{PSL}_2(\mathfrak{a})$*, where* $\mathfrak{a}$ *is a fractional ideal of an imaginary quadratic field* K*,*

Almost every admissible number is the bend of a circle in $\mathcal{P}$*. Quantitatively, the number of exceptions up to* N *is bounded by* $O(N^{1-\eta})$*.*

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Theorem (Fuchs–Stange–Zhang, 2019)

Suppose that

- $\mathcal{P}$ *is a primitive integral Kleinian* 1*-sphere packing in* $\widehat{\mathbb{R}^2}$ *with an automorphism group* $\Gamma < \mathrm{PSL}_2(\mathfrak{a})$*, where* $\mathfrak{a}$ *is a fractional ideal of an imaginary quadratic field* K*,*
- *there exist circles* S_0 *and* S_1 *such that* S_1 *is in* $\mathcal{P}$ *and is tangent to* S_0*, and*

Almost every admissible number is the bend of a circle in $\mathcal{P}$*. Quantitatively, the number of exceptions up to* N *is bounded by* $O(N^{1-\eta})$*.*

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Almost every admissible number is the bend of a circle in $\Gamma \cdot S_1 \subseteq \mathcal{P}$*. Quantitatively, the number of exceptions up to* N *is bounded by* $O(N^{1-\eta})$*.*

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Result for Kleinian 1-Sphere (Circle) Packings

Theorem (Fuchs–Stange–Zhang, 2019)

If $\mathcal{P}$ is a primitive integral Kleinian 1-sphere packing in $\widehat{\mathbb{R}^2}$ satisfying certain conditions, almost every admissible number is the bend of a circle in $\mathcal{P}$.

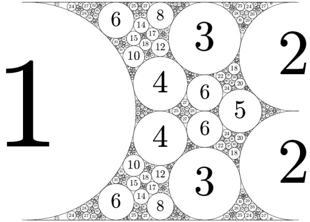

Figure: An integral Kleinian (more specifically, cuboctahedral) 1-sphere packing that satisfies the conditions of the theorem. Figure taken from "Local-Global Principles in Circle Packings" by Fuchs, Stange, and Zhang.

Proof Outline

1. The assumption that the stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ implies that the set of bends of $\mathcal{P}$ contains primitive values of a shifted binary quadratic form.

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1. The assumption that the stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ implies that the set of bends of $\mathcal{P}$ contains primitive values of a shifted binary quadratic form.
2. The shifted binary quadratic form gives you enough to work with so that you can apply the circle method (with some other tools, including spectral theory, used in the major arc and minor arc analyses) to obtain an "almost all" statement.

My Research

Goal: Prove an asymptotic local-global principle for bends of certain integral Kleinian sphere packings.

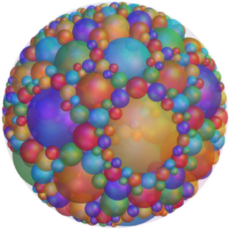

Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing made by Kei Nakamura.

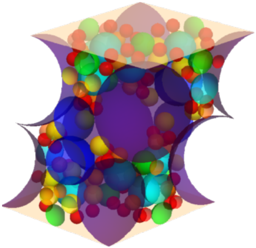

Figure: A fundamental domain of an integral Kleinian sphere packing made by Arseniy (Senia) Sheydvasser.

Theorem in Progress

Suppose that

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- $\mathcal{P}$ is a primitive integral Kleinian $(n-1)$-sphere packing in $\widehat{\mathbb{R}^n}$ with an automorphism group Γ of Möbius transformations,

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Then every sufficiently large admissible integer is a bend of a $(n-1)$-sphere in $\Gamma \cdot S_1 \subseteq \mathcal{P}$.

Proof Methods Outline

1. The assumption that the stabilizer of S_0 in Γ contains (up to conjugacy) a congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ implies that the set of bends of $\mathcal{P}$ contains "primitive" values of a shifted quaternary quadratic form.

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2. This shifted quaternary quadratic form should give you enough to work with so that you can apply the circle method to show that every sufficiently large admissible number is represented as a bend.

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   - Major arcs (for main term that contains local information): use spectral theory and expander graphs
   - Minor arcs (for error term): use Kloosterman circle method

Quadratic Form for Soddy Sphere Packings

Example (Quadratic form for Soddy sphere packings)

Shifted quaternary quadratic form in $a_0, a_1, c_0, c_1 \in \mathbb{Z}$:

$$\hat{\beta}|C(a_0 + a_1\omega) + D\varrho(c_0 + c_1\omega)|^2 - Dj\bar{C} + Cj\bar{D},$$

$\omega = e^{\pi i/3}$
$\varrho = 1 + \omega$
$\gcd(a_0 + a_1\omega, \varrho(c_0 + c_1\omega)) = 1$

$\hat{\beta} \in \mathbb{R}$ and $C, D \in \mathbb{H}$ only depend on the packing.

(Scale appropriately to obtain a primitive integral quadratic form.)

Obtained by examining how Möbius transformations affect bends of $(n-1)$-spheres

Illustrations Credits

Besides the illustrations previously credited and a few Apollonian circle packing construction illustrations created by the presenter, the illustrations for this talk came from the following papers:

- Jean Bourgain and Alex Kontorovich, "On the local-global conjecture for integral Apollonian gaskets," *Inventiones mathematicae*, volume 196, pp. 589–650, 2014.
- Alex Kontorovich, "From Apollonius to Zaremba: Local-global phenomena in thin orbits," *Bulletin of the American Mathematical Society*, volume 50, number 2, pp. 187-228, 2013, `https://www.ams.org/journals/bull/2013-50-02/S0273-0979-2013-01402-2/`.
- Alex Kontorovich, "The Local-Global Principle for Integral Soddy Sphere Packings," *Journal of Modern Dynamics*, volume 15, pp. 209-236, 2019, `https://www.aimsciences.org/article/doi/10.3934/jmd.2019019`.

Thank you for listening!