

On the local-global conjecture for higher-dimensional Kleinian sphere packings

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Packings and Beyond
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Soddy sphere packings: The construction

Given four pairwise tangent spheres with disjoint points of tangency, there are exactly two spheres tangent to the given ones.

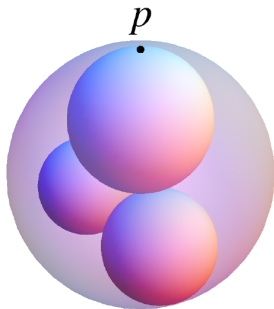


Figure: Four pairwise tangent spheres.

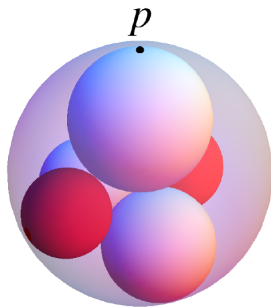


Figure: Four pairwise tangent spheres with two additional tangent spheres.

Soddy sphere packings: The construction

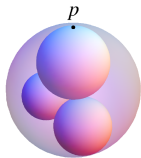


Figure: Four
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Soddy sphere packings: The construction

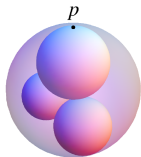


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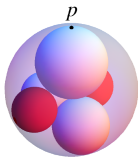


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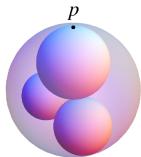


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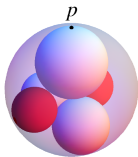


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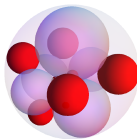


Figure: More tangent spheres.

Soddy sphere packings: The construction

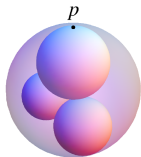


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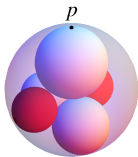


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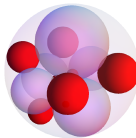


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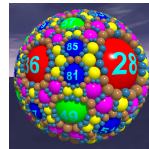


Figure: A Soddy sphere packing.

Soddy sphere packings and the integers

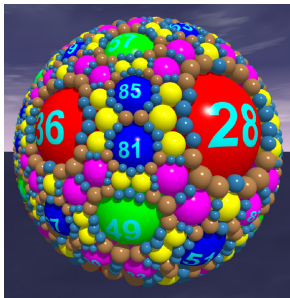


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

Label on sphere:
 $\text{bend} = 1/\text{radius}$

Soddy sphere packings and the integers

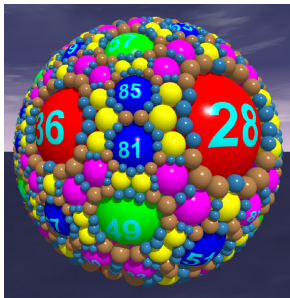


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All of the bends of this Soddy sphere packing are integers.

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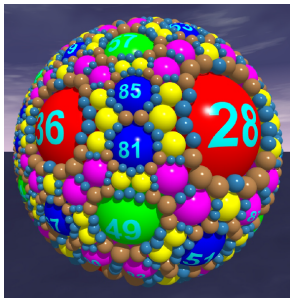


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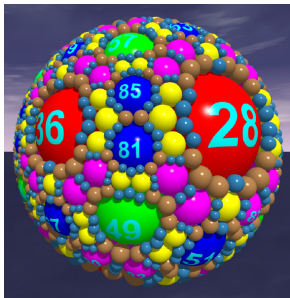


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Label on sphere:
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All of the bends of this Soddy sphere packing are integers.

Which integers appear as bends?

Are there any congruence or local obstructions?

Definition (Admissible integers for Soddy sphere packings)

Let \mathcal{P} be an integral Soddy sphere packing.

An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some sphere in } \mathcal{P} \pmod{q}.$$

Equivalently, m is admissible if m has no local obstructions.

Admissible integers

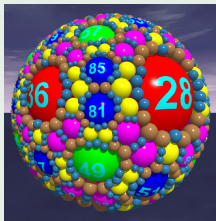
Theorem (Kontorovich, 2019)

m is admissible in a primitive integral Soddy sphere packing \mathcal{P} if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod{3},$$

where $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ depends only on the packing.

Example



m is admissible \iff
 $m \equiv 0 \text{ or } 1 \pmod{3}.$

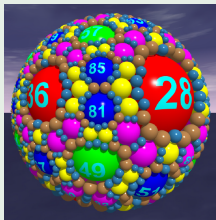
The (strong asymptotic) local-global theorem for Soddy sphere packings

Theorem (Kontorovich, 2019)

The bends of a fixed primitive integral Soddy sphere packing \mathcal{P} satisfy a (strong asymptotic) local-global principle.

That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a sphere in the packing.

Example



If $m \equiv 0$ or $1 \pmod{3}$ and m is sufficiently large, then m is the bend of a sphere in the packing.

Proof outline for Soddy sphere packing result

- 1 Show that the automorphism/symmetry group of the Soddy sphere packing contains a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains “primitive” values of a quaternary quadratic polynomial.

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- 2 The quaternary quadratic polynomial gives you enough to work with so that you can quote the result of the circle method to give an asymptotic formula involving a singular series.
- 3 Show that the singular series (with the primitivity restriction) is bounded away from zero when m is admissible.

Congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$

Definition (Principal congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$)

For an imaginary quadratic field K , a **principal congruence subgroup** of $\mathrm{PSL}_2(\mathcal{O}_K)$ is a subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ of the form

$$\Lambda(\varrho) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varrho} \right\}$$

for a fixed element ϱ of \mathcal{O}_K .

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Example (Soddy sphere packing, Kontorovich, 2019)

There exists a sphere $S_0 \in \mathcal{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) : b, c \equiv 0 \pmod{\varrho} \right\},$$

where $\mathcal{O} = \mathbb{Z}[e^{\pi i/3}]$ and $\varrho = 1 + e^{\pi i/3}$.

Kleinian sphere packings

Definition (Kleinian sphere packing)

An $(n - 1)$ -sphere packing \mathcal{P} is **Kleinian** if its limit set is that of a geometrically finite group $\Gamma < \text{Isom}(\mathcal{H}^{n+1})$.

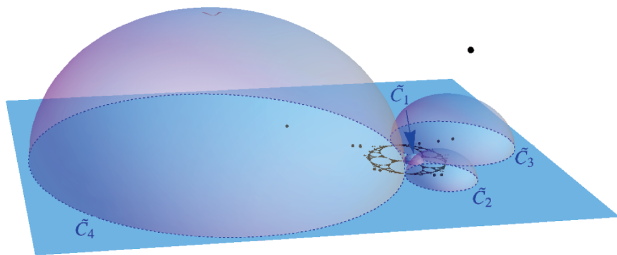


Figure: Apollonian circle packing as the limit set of Γ . Image by Alex Kontorovich.

Definition (Kleinian sphere packing)

An $(n - 1)$ -sphere packing \mathcal{P} is **Kleinian** if its limit set is that of a geometrically finite group $\Gamma < \text{Isom}(\mathcal{H}^{n+1})$.

- Action of $\text{Isom}(\mathcal{H}^{n+1})$ extends continuously to $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, the boundary of \mathcal{H}^{n+1} .
- Γ stabilizes \mathcal{P} (i.e., Γ maps \mathcal{P} to itself).
- Γ is a thin group.

Examples of integral Kleinian sphere packings

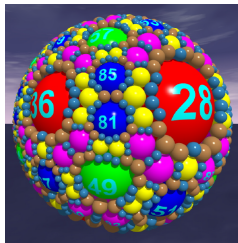


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

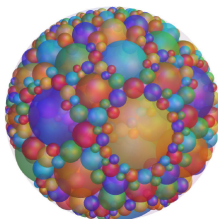


Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

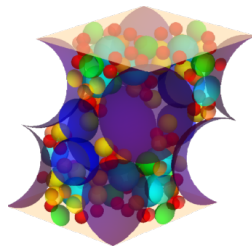


Figure: A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.

(Strong asymptotic) local-global principles

Goal: Prove (strong asymptotic) local-global principles for certain integral Kleinian sphere packings, that is, prove:

If m is admissible and sufficiently large, then m is the bend of an $(n - 1)$ -sphere in the packing.

Definition (Admissible integers)

Let \mathcal{P} be an integral Kleinian sphere packing.

An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some } (n - 1)\text{-sphere in } \mathcal{P} \pmod{q}.$$

Why should we have (strong asymptotic) local-global principles?

Theorem (Kim, 2015)

Let \mathcal{P} be a Kleinian $(n - 1)$ -sphere packing with $n \geq 2$.

The number of spheres in \mathcal{P} with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^δ , where $\delta =$ the Hausdorff dimension of the closure of \mathcal{P} .

For us,

$$\delta > n - 1 \geq 2.$$

Thus, we would expect that the multiplicity of a given admissible bend up to N is roughly $N^{\delta-1} \gg N$, so we should expect that every sufficiently large admissible number to be represented.

Progress towards (strong asymptotic) local-global conjectures for Kleinian sphere packings

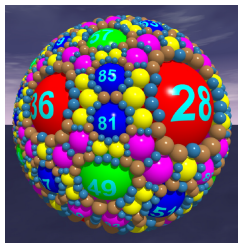


Figure: (Strong asymptotic) local-global principle proven for Soddy sphere packings by Alex Kontorovich in 2019 (arXiv 2012).

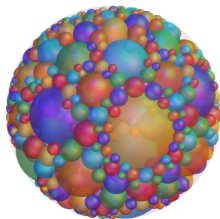
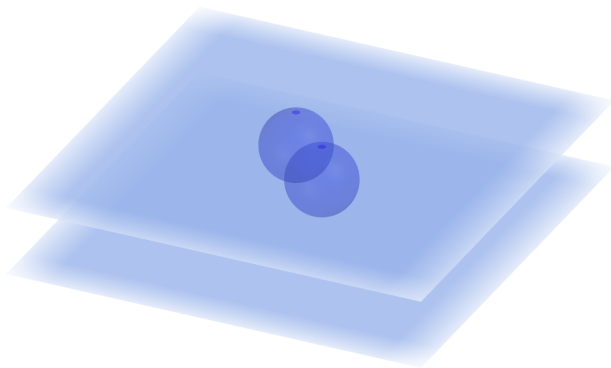


Figure: Partial local-global results for orthoplicial sphere packings independently proven by Kei Nakamura (arXiv 2014) and Dimitri Dias (arXiv 2014).

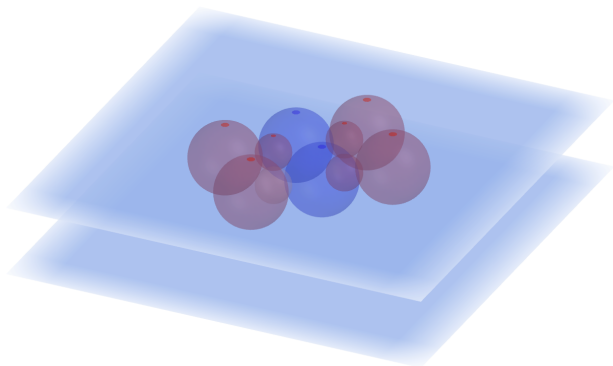
Orthoplicial sphere packings: The construction

Four pairwise tangent spheres form two gaps. In each gap, there is a unique way to inscribe four pairwise tangent spheres such that each inscribed sphere is tangent to exactly three of the original spheres.



Orthoplicial sphere packings: The construction

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Orthoptical sphere packings: The construction



Figure: An orthoptical sphere octuple.

Orthoptical sphere packings: The construction



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Figure: Adding more spheres.

Orthoptical sphere packings: The construction



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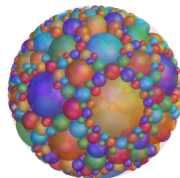


Figure: An integral orthoptical sphere packing.

A congruence restriction for orthoptical sphere packings

Theorem (Nakamura, 2014; Dias, 2014)

For a primitive orthoptical sphere packing \mathcal{P} , there exists $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ such that every bend b of a sphere in \mathcal{P} satisfies

$$b \equiv 0, 2, \text{ or } \varepsilon(\mathcal{P}) \pmod{4}.$$

Example



Every bend in this packing is congruent to 0, 1, or 2 (mod 4).

Theorem (Dias, 2014)

Let a and b be the bends of two tangent spheres in a primitive orthoplicial sphere packing \mathcal{P} such that a is nonzero and even and b is odd.

Every sufficiently large integer m that satisfies $\gcd(m, a) = 1$ is the bend of a sphere in \mathcal{P} if and only if $m \equiv b \pmod{4}$.

Proof methods similar to those in Kontorovich's paper on the local-global principle for Soddy sphere packings.

Partial local-global principle for orthoplicial sphere packings

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Proof methods similar to those in Kontorovich's paper on the local-global principle for Soddy sphere packings.

Using essentially the same proofs as Dias, we have the following.

Theorem (J., 2025+)

Let a and b be the bends of two tangent spheres in a primitive orthoplicial sphere packing \mathcal{P} such that $a + b$ is odd and $a \neq 0$.

Every sufficiently large integer m that satisfies $\gcd(m, a) = 1$ and $m \equiv b \pmod{4}$ is the bend of a sphere in \mathcal{P} .

Partial local-global principle for orthoptical sphere packings

Theorem (J., 2025+)

Let a and b be the bends of two tangent spheres in a primitive orthoptical sphere packing \mathcal{P} such that $a + b$ is odd and $a \neq 0$. Every sufficiently large integer m that satisfies $\gcd(m, a) = 1$ and $m \equiv b \pmod{4}$ is the bend of a sphere in \mathcal{P} .

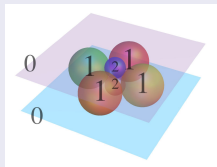
Example



For this orthoptical sphere packing, every sufficiently large m that satisfies $\gcd(m, -7) = 1$ and $m \equiv 12 \equiv 0 \pmod{4}$ is the bend of a sphere in this packing.

Partial local-global principle for orthoplicial sphere packings

Corollary (J., 2025+)



Let \mathcal{P}_0 be the orthoplicial sphere packing generated by the octuple on the left. Every sufficiently large integer m that is $m \equiv 0, 1, \text{ or } 2 \pmod{4}$ is the bend of a sphere in \mathcal{P}_0 .

Corollary (J., 2025+)



Let \mathcal{P}_1 be the orthoplicial sphere packing generated by the octuple on the left. Every sufficiently large integer m that is $m \equiv -1, 0, \text{ or } 2 \pmod{4}$ is the bend of a sphere in \mathcal{P}_1 .

I am working on (strong asymptotic) local-global principles for certain integral Kleinian sphere packings.

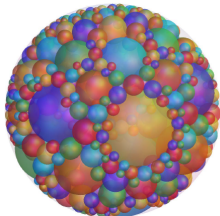


Figure: An integral orthoplicial sphere packing. Image by Kei Nakamura.

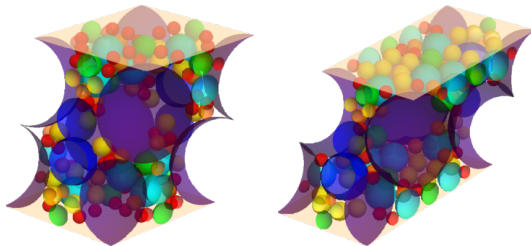


Figure: A fundamental domains of two conformally inequivalent integral Kleinian sphere packing. Images by Arseniy (Senia) Sheydvasser.

Besides the illustrations previously credited and a few orthoplicial octuple illustrations created by the presenter, the illustrations for this talk came from the following papers:

- Alex Kontorovich, “The Local-Global Principle for Integral Soddy Sphere Packings,” *Journal of Modern Dynamics*, volume 15, pp. 209-236, 2019, <https://www.aims sciences.org/article/doi/10.3934/jmd.2019019>
- Kei Nakamura, “The local-global principle for integral bends in orthoplicial Apollonian sphere packings,” preprint, arXiv:1401.2980

Thank you for listening!

