On the local-global conjecture for higher-dimensional Kleinian sphere packings

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Soddy sphere packings: The construction

Given four pairwise tangent spheres with disjoint points of tangency, there are exactly two spheres tangent to the given ones.

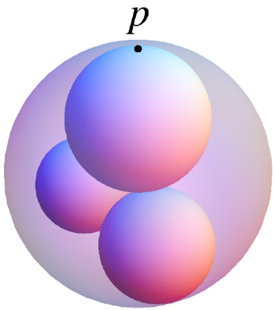

Figure: Four pairwise tangent spheres.

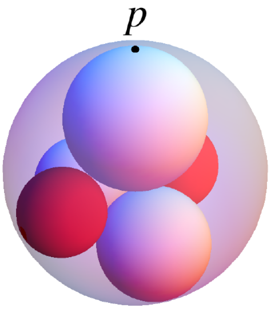

Figure: Four pairwise tangent spheres with two additional tangent spheres.

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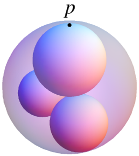

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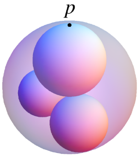

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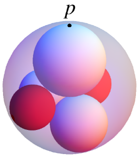

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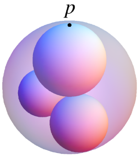

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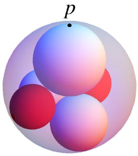

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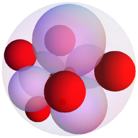

Figure: More tangent spheres.

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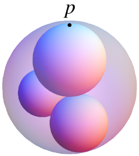

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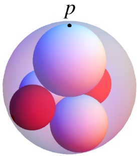

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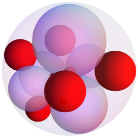

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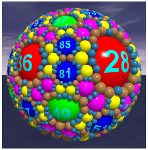


Figure: A Soddy sphere packing.

Soddy sphere packings and the integers

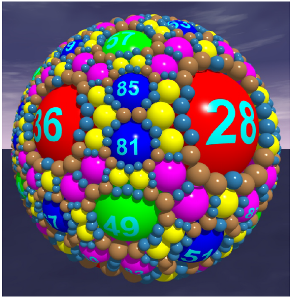
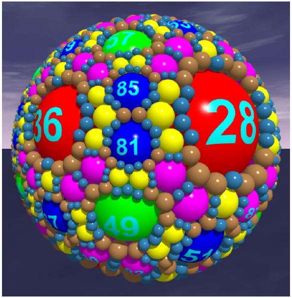


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

Label on sphere:

bend $= 1/\text{radius}$



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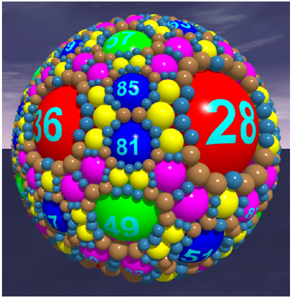

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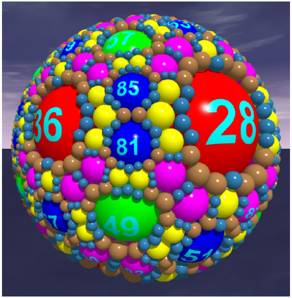

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Label on sphere:
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Which integers appear as bends?

Are there any congruence or local obstructions?

Admissible integers

Definition (Admissible integers for Soddy sphere packings)

Let $\mathcal{P}$ be an integral Soddy sphere packing.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some sphere in } \mathcal{P} \pmod{q}.$$

Equivalently, m is admissible if m has no local obstructions.

Admissible integers

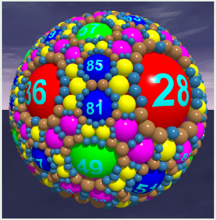
Theorem (Kontorovich, 2019)

m is admissible in a primitive integral Soddy sphere packing $\mathcal{P}$ if and only if

$$m \equiv 0 \text{ or } \varepsilon(\mathcal{P}) \pmod 3,$$

where $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ depends only on the packing.

Example



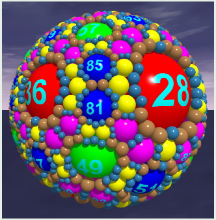
m is admissible $\Longleftrightarrow$
$m \equiv 0$ or $1 \pmod 3$.

The (strong asymptotic) local-global theorem for Soddy sphere packings

Theorem (Kontorovich, 2019)

The bends of a fixed primitive integral Soddy sphere packing $\mathcal{P}$ satisfy a (strong asymptotic) local-global principle.
That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a sphere in the packing.

Example



If $m \equiv 0$ or $1 \pmod 3$ and m is sufficiently large, then m is the bend of a sphere in the packing.

Proof outline for Soddy sphere packing result

1. Show that the automorphism/symmetry group of the Soddy sphere packing contains a congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z}[e^{\pi i/3}])$, and this congruence subgroup maps a particular sphere to itself. This implies that the set of bends contains "primitive" values of a quaternary quadratic polynomial.

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2. The quaternary quadratic polynomial gives you enough to work with so that you can quote the result of the circle method to give an asymptotic formula involving a singular series.
3. Show that the singular series (with the primitivity restriction) is bounded away from zero when m is admissible.

Congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$

Definition (Principal congruence subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$)

For an imaginary quadratic field K, a **principal congruence subgroup** of $\mathrm{PSL}_2(\mathcal{O}_K)$ is a subgroup of $\mathrm{PSL}_2(\mathcal{O}_K)$ of the form

$$\Lambda(\varrho) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_K) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\varrho} \right\}$$

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Example (Soddy sphere packing, Kontorovich, 2019)

There exists a sphere $S_0 \in \mathcal{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) the congruence subgroup

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) : b, c \equiv 0 \pmod{\varrho} \right\},$$

where $\mathcal{O} = \mathbb{Z}[e^{\pi i/3}]$ and $\varrho = 1 + e^{\pi i/3}$.

Proof sketch for Soddy sphere packing result

Let b_1, b_2, b_3, b_4, and b_5 be bends of five pairwise tangent spheres in $\mathcal{P}$. Let $\mathcal{O} = \mathbb{Z}[e^{\pi i/3}]$ and $\varrho = 1 + e^{\pi i/3}$.
Having $S_0 \in \mathcal{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) the congruence subgroup

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}) : \beta, \gamma \equiv 0 \pmod{\varrho} \right\},$$

$\implies$ When $\varrho\gamma = \varrho(\gamma_1 + \gamma_2 e^{\pi i/3})$ and $\delta = \delta_1 + \delta_2 e^{\pi i/3}$ are coprime in $\mathcal{O}$,

$$\begin{aligned} F(\varrho\gamma, \delta) = 3A(\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2) + 3B(\gamma_1\delta_1 + \gamma_2\delta_2) - 3C\gamma_2\delta_1 \\ + 3(B+C)\gamma_1\delta_2 + D(\delta_1^2 + \delta_1\delta_2 + \delta_2^2) - b_1 \end{aligned}$$

is a bend in $\mathcal{P}$, where

$$A = b_1 + b_4, \qquad B = \frac{b_1 + b_2 - 2b_3 + b_4 + b_5}{3},$$

$$C = \frac{b_1 - 2b_2 + b_3 + b_4 + b_5}{3}, \qquad D = b_1 + b_5.$$

Proof sketch for Soddy sphere packing result (cont.)

Have quaternary quadratic form

$$f(\varrho\gamma, \delta) = F(\varrho\gamma, \delta) + b_1$$

with $\varrho\gamma\mathcal{O} + \delta\mathcal{O} = \mathcal{O}$.

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$$R(n) = \sum_{\substack{\gamma,\delta\in\mathcal{O} \\ \varrho\gamma\mathcal{O}+\delta\mathcal{O}=\mathcal{O}}} \mathbf{1}_{\{F(\varrho\gamma,\delta)=n+b_1\}} = \sum_{\substack{\gamma,\delta\in\mathcal{O} \\ \varrho\gamma\mathcal{O}+\delta\mathcal{O}=\mathcal{O}}} \mathbf{1}_{\{f(\varrho\gamma,\delta)=n\}},$$

where

$$\mathbf{1}_{\{Y\}} = \begin{cases} 1 & \text{if } Y \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

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Want $R(n) > 0$ when $n + b_1$ is admissible.

Proof sketch for Soddy sphere packing result (cont.)

To analyze $R(n)$,

- use inclusion-exclusion to remove coprimality condition $(\varrho\gamma\mathcal{O} + \delta\mathcal{O} = \mathcal{O})$,

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- analyze a singular series (which contains local information and involves a product of local densities of solutions).

Kleinian sphere packings

Definition (Kleinian sphere packing)

An $(n-1)$-sphere packing $\mathcal{P}$ is **Kleinian** if its limit set is that of a geometrically finite group $\Gamma < \mathrm{Isom}(\mathcal{H}^{n+1})$.

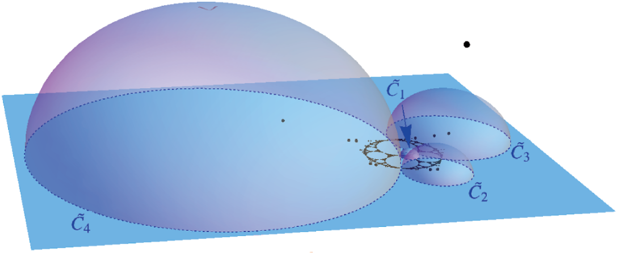

Figure: Apollonian circle packing as the limit set of Γ. Image by Alex Kontorovich.

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- Action of $\mathrm{Isom}(\mathcal{H}^{n+1})$ extends continuously to $\widehat{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, the boundary of $\mathcal{H}^{n+1}$.
- Γ stabilizes $\mathcal{P}$ (i.e., Γ maps $\mathcal{P}$ to itself).
- Γ is a thin group.

Examples of integral Kleinian sphere packings

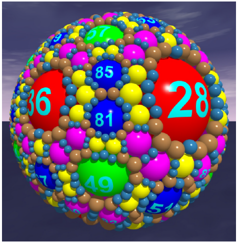


Figure: An integral Soddy sphere packing. Image by Nicolas Hannachi.

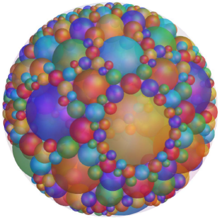


Figure: An integral Kleinian (more specifically, an orthoplicial) sphere packing. Image by Kei Nakamura.

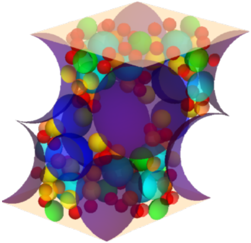


Figure: A fundamental domain of an integral Kleinian sphere packing. Image by Arseniy (Senia) Sheydvasser.

(Strong asymptotic) local-global principles

Goal: Prove (strong asymptotic) local-global principles for certain integral Kleinian sphere packings, that is, prove:
If m is admissible and sufficiently large, then m is the bend of an $(n-1)$-sphere in the packing.

Definition (Admissible integers)

Let $\mathcal{P}$ be an integral Kleinian sphere packing.
An integer m is **admissible (or locally represented)** if for every $q \geq 1$

$$m \equiv \text{bend of some } (n-1)\text{-sphere in } \mathcal{P} \pmod{q}.$$

Why should we have (strong asymptotic) local-global principles?

Theorem (Kim, 2015)

Let $\mathcal{P}$ be a Kleinian $(n-1)$-sphere packing with $n \geq 2$. The number of spheres in $\mathcal{P}$ with bend at most N (counted with multiplicity) is asymptotically equal to a constant times N^{δ}, where $\delta =$ the Hausdorff dimension of the closure of $\mathcal{P}$.

For us,

$$\delta > n - 1 \geq 2.$$

Thus, we would would expect that the multiplicity of a given admissible bend up to N is roughly $N^{\delta-1} \gg N$, so we should expect that every sufficiently large admissible number to be represented.

Progress towards (strong asymptotic) local-global conjectures for Kleinian sphere packings

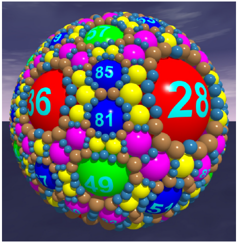


Figure: (Strong asymptotic) local-global principle proven for Soddy sphere packings by Alex Kontorovich in 2019 (arXiv 2012).

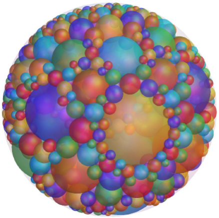
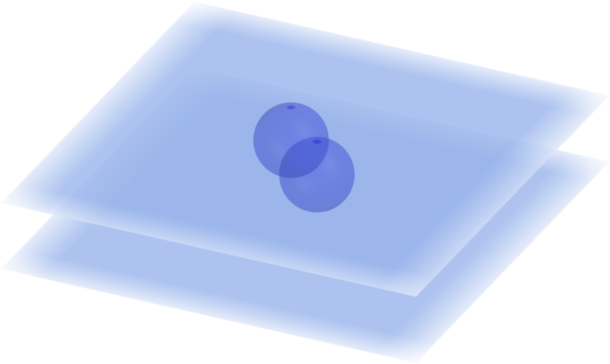


Figure: Partial local-global results for orthoplicial sphere packings independently proven by Kei Nakamura (arXiv 2014) and Dimitri Dias (arXiv 2014).

Orthoplicial sphere packings: The construction

Four pairwise tangent spheres form two gaps. In each gap, there is a unique way to inscribe four pairwise tangent spheres such that each inscribed sphere is tangent to exactly three of the original spheres.



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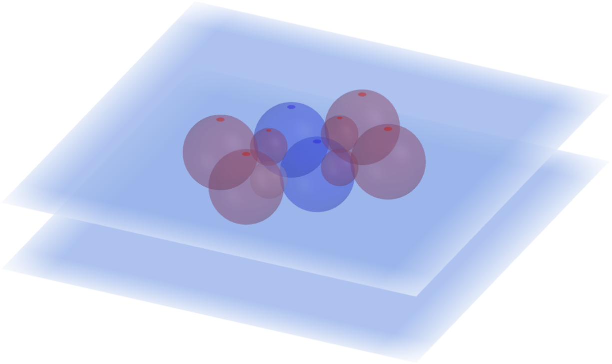

Orthoplicial sphere packings: The construction

Figure: An orthoplicial sphere octuple.

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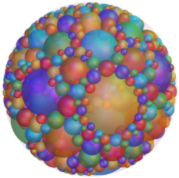

Figure: An integral orthoplicial sphere packing.

A congruence restriction for orthoplicial sphere packings

Theorem (Nakamura, 2014; Dias, 2014)

For a primitive orthoplicial sphere packing $\mathcal{P}$, there exists $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ such that the bend b of any sphere in $\mathcal{P}$ satisfies

$$b \equiv 0, 2, \textit{or } \varepsilon(\mathcal{P}) \pmod 4.$$

Example

Every bend in this packing is congruent to $0, 1,$ or $2 \pmod 4$.

Partial local-global principle for orthoplicial sphere packings

Theorem (Dias, 2014)

Let a and b be the bends of two tangent spheres in a primitive orthoplicial sphere packing $\mathcal{P}$ such that a is nonzero and even and b is odd.
Every sufficiently large integer m that satisfies $\gcd(m, a) = 1$ is the bend of a sphere in $\mathcal{P}$ if and only if $m \equiv b \pmod 4$.

Proof methods similar to those in Kontorovich's paper on the local-global principle for Soddy sphere packings.

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Proof methods similar to those in Kontorovich's paper on the local-global principle for Soddy sphere packings.
There exists a sphere $S_0 \in \mathcal{P}$ such that the stabilizer of S_0 in Γ contains (up to conjugacy) the congruence subgroup

$$\left\{ g \in \mathrm{PSL}_2(\mathbb{Z}[i]) : g \equiv \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ or } \begin{pmatrix} \pm i & 0 \\ 0 & \mp i \end{pmatrix} \pmod 2 \right\},$$

This gives rise to a somewhat similar quadratic polynomial for bends.

Partial local-global principle for orthoplicial sphere packings

Using essentially the same proofs as Dias, we have the following.

Theorem (J., 2025+)

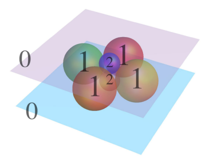
Let a and b be the bends of two tangent spheres in a primitive orthoplicial sphere packing $\mathcal{P}$ such that $a+b$ is odd and $a \neq 0$. Every sufficiently large integer m that satisfies $\gcd(m,a)=1$ and $m \equiv b \pmod 4$ is the bend of a sphere in $\mathcal{P}$.

Example

For this orthoplicial sphere packing, every sufficiently large m that satisfies $\gcd(m,-7)=1$ and $m \equiv 12 \equiv 0 \pmod 4$ is the bend of a sphere in this packing.

Partial local-global principle for orthoplicial sphere packings

Corollary (J., 2025+)



Let $\mathcal{P}_0$ be the orthoplicial sphere packing generated by the octuple on the left. Every sufficiently large integer m that is $m \equiv 0, 1,$ or $2 \pmod 4$ is the bend of a sphere in $\mathcal{P}_0$.

Corollary (J., 2025+)

Let $\mathcal{P}_1$ be the orthoplicial sphere packing generated by the octuple on the left. Every sufficiently large integer m that is $m \equiv -1, 0,$ or $2 \pmod 4$ is the bend of a sphere in $\mathcal{P}_1$.

Proof of local-global principle for particular orthoplicial sphere packing

Corollary (J., 2025+)

Let $\mathcal{P}_1$ be the orthoplicial sphere packing generated by the octuple on the left. Every sufficiently large integer m that is $m \equiv -1, 0,$ or $2 \pmod 4$ is the bend of a sphere in $\mathcal{P}_1$.

Proof. -1, 2, and 4 are bends of 3 pairwise tangent spheres in $\mathcal{P}_1$. $-1+2$ and $-1+4$ are odd, so every sufficiently large integer m that satisfies one of the following is the bend of a sphere in $\mathcal{P}_1$.

1. $\gcd(m, 2) = 1$ and $m \equiv -1 \pmod 4$
2. $\gcd(m, -1) = 1$ and $m \equiv 4 \equiv 0 \pmod 4$
3. $\gcd(m, -1) = 1$ and $m \equiv 2 \pmod 4$

□

My research

I am working on (strong asymptotic) local-global principles for certain integral Kleinian sphere packings.

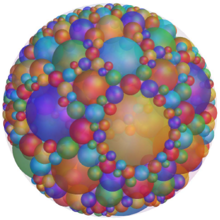


Figure: An integral orthoplicial sphere packing. Image by Kei Nakamura.

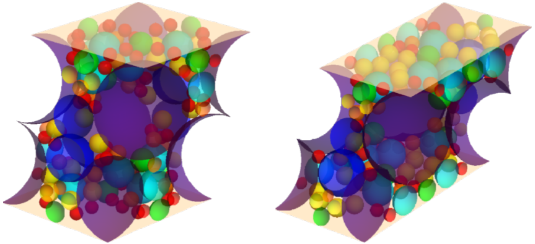


Figure: A fundamental domains of two conformally inequivalent integral Kleinian sphere packing. Images by Arseniy (Senia) Sheydvasser.

Illustrations credits

Besides the illustrations previously credited and a few orthoplicial octuple illustrations created by the presenter, the illustrations for this talk came from the following papers:

- Alex Kontorovich, "The Local-Global Principle for Integral Soddy Sphere Packings," *Journal of Modern Dynamics*, volume 15, pp. 209-236, 2019, `https://www.aimsciences.org/article/doi/10.3934/jmd.2019019`
- Kei Nakamura, "The local-global principle for integral bends in orthoplicial Apollonian sphere packings," preprint, arXiv:1401.2980

Thank you for listening!