

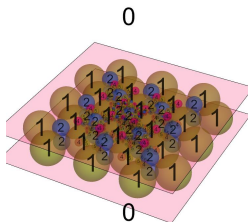
The strong asymptotic local-to-global principle for orthoplicial sphere packings

Edna Jones

Tulane University

Special Session on Recent Developments in Number Theory
AMS 2026 Fall Southeastern Sectional Meeting
October 11, 2026

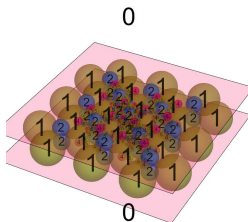
Orthohpplcial sphere packings and the integers



Label on sphere:
 $\text{bend} = 1/(\text{signed radius})$



Orthohpplcial sphere packings and the integers



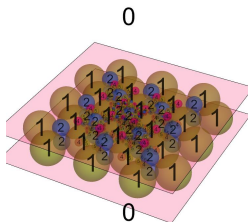
Label on sphere:

$$\text{bend} = 1/(\text{signed radius})$$

All of the bends of spheres in these orthohpplcial sphere packings are integers.



Orthohpplcial sphere packings and the integers



Label on sphere:

$$\text{bend} = 1/(\text{signed radius})$$

All of the bends of spheres in these orthohpplcial sphere packings are integers.



Which integers appear as bends?

Orthoplicial sphere packings: The construction

Given four pairwise tangent spheres, there are exactly two ways to inscribe four pairwise tangent spheres such that each inscribed sphere is tangent to exactly three of the original spheres.

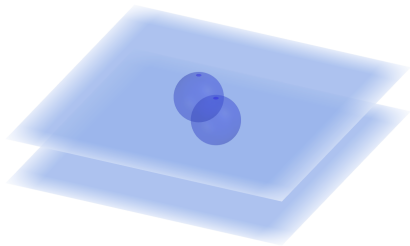


Figure: Four pairwise tangent generalized spheres.

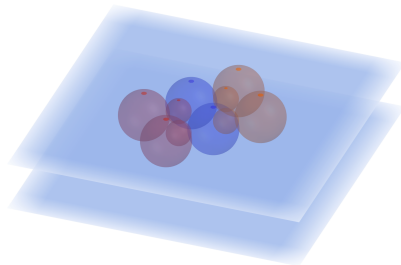


Figure: Four pairwise tangent generalized spheres with eight additional spheres.

Orthoplicial sphere packings: The construction



Figure: An orthoplicial sphere octuple.

Orthoplicial sphere packings: The construction



Figure: An orthoplicial sphere octuple.



Figure: Adding more spheres.

Orthoplicial sphere packings: The construction



Figure: An orthoplicial sphere octuple.



Figure: Adding more spheres.

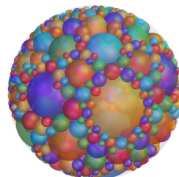
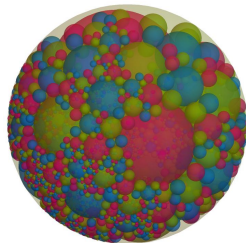
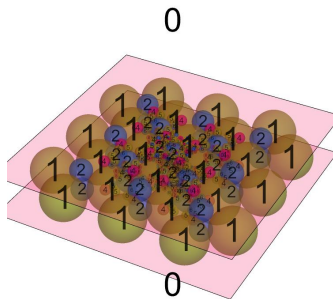


Figure: An integral orthoplicial sphere packing.

Examples of integral orthoptical sphere packings



Label on sphere: $\text{bend} = 1/(\text{signed radius})$

- $\text{bend} \equiv 0 \pmod{4}$
- $\text{bend} \equiv 1 \pmod{4}$
- $\text{bend} \equiv 2 \pmod{4}$
- $\text{bend} \equiv 3 \pmod{4}$

A congruence restriction for orthoplicial sphere packings

Theorem (Nakamura, 2014; Dias, 2014)

For a primitive integral orthoplicial sphere packing $\mathcal{P}$, there exists $\varepsilon(\mathcal{P}) \in \{\pm 1\}$ such that the bend b of any sphere in $\mathcal{P}$ satisfies

$$b \equiv 0, 2, \text{ or } \varepsilon(\mathcal{P}) \pmod{4}.$$

Example



Every bend in this packing is congruent to 0, 1, or 2 (mod 4).

Partial local-to-global principle for orthoplicial sphere packings

Theorem (Dias, 2014)

Let b_1 and b_2 be the bends of two tangent spheres in a primitive integral orthoplicial sphere packing $\mathcal{P}$ such that b_1 is nonzero and even and b_2 is odd.

Every sufficiently large integer m that satisfies $\gcd(m, b_1) = 1$ is the bend of a sphere in $\mathcal{P}$ if and only if $m \equiv b_2 \pmod{4}$.

Partial local-to-global principle for orthoplicial sphere packings

Theorem (Dias, 2014)

Let b_1 and b_2 be the bends of two tangent spheres in a primitive integral orthoplicial sphere packing $\mathcal{P}$ such that b_1 is nonzero and even and b_2 is odd.

Every sufficiently large integer m that satisfies $\gcd(m, b_1) = 1$ is the bend of a sphere in $\mathcal{P}$ if and only if $m \equiv b_2 \pmod{4}$.

Theorem (J., 2026+)

Let b_1 and b_2 be the bends of two tangent spheres in a primitive integral orthoplicial sphere packing $\mathcal{P}$ such that $b_1 + b_2$ is odd and $b_1 \neq 0$.

Every sufficiently large integer $m \equiv b_2 \pmod{4}$ is the bend of a sphere in $\mathcal{P}$.

Strong asymptotic local-to-global principle for orthoplicial sphere packings

Theorem (J., 2026+)

Let b_1 and b_2 be the bends of two tangent spheres in a primitive integral orthoplicial sphere packing $\mathcal{P}$ such that $b_1 + b_2$ is odd and $b_1 \neq 0$.

Every sufficiently large integer $m \equiv b_2 \pmod{4}$ is the bend of a sphere in $\mathcal{P}$.

Strong asymptotic local-to-global principle for orthoplicial sphere packings

Theorem (J., 2026+)

Let b_1 and b_2 be the bends of two tangent spheres in a primitive integral orthoplicial sphere packing $\mathcal{P}$ such that $b_1 + b_2$ is odd and $b_1 \neq 0$.

Every sufficiently large integer $m \equiv b_2 \pmod{4}$ is the bend of a sphere in $\mathcal{P}$.

Corollary (J., 2026+)

The bends of a fixed primitive integral orthoplicial sphere packing $\mathcal{P}$ satisfy a strong asymptotic local-to-global principle.

That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m meets the congruence restrictions for the packing, then m is the bend of a sphere in the packing.

Strong asymptotic local-to-global principle for orthopical sphere packings

Corollary (J., 2026+)

The bends of a fixed primitive integral orthopical sphere packing $\mathcal{P}$ satisfy a strong asymptotic local-to-global principle.

That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m meets the congruence restrictions for the packing, then m is the bend of a sphere in the packing.

Example



In this orthopical sphere packing, every sufficiently large $m \equiv 0, 1, \text{ or } 2 \pmod{4}$ is the bend of a sphere in this packing.

Strong asymptotic local-to-global principle for orthoplicial sphere packings

Corollary (J., 2026+)

The bends of a fixed primitive integral orthoplicial sphere packing $\mathcal{P}$ satisfy a strong asymptotic local-to-global principle.

That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m meets the congruence restrictions for the packing, then m is the bend of a sphere in the packing.

Example



Let $\mathcal{P}_1$ be the orthoplicial sphere packing generated by the octuple on the left. Every sufficiently large integer $m \equiv -1, 0$, or $2 \pmod{4}$ is the bend of a sphere in $\mathcal{P}_1$.

- 1 Show that the automorphism/symmetry group of the orthoplicial sphere packing contains

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}[i]) : b, c \in 2\mathbb{Z}[i] \right\},$$

and this subgroup maps a particular sphere to itself. This implies that the set of bends contains “primitive” values of a quaternary quadratic polynomial.

- 1 Show that the automorphism/symmetry group of the orthoplicial sphere packing contains

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}[i]) : b, c \in 2\mathbb{Z}[i] \right\},$$

and this subgroup maps a particular sphere to itself. This implies that the set of bends contains “primitive” values of a quaternary quadratic polynomial.

- 2 The quaternary quadratic polynomial gives you enough to work with so that you can use the circle method to give an asymptotic formula involving a singular series.

- 1 Show that the automorphism/symmetry group of the orthoplicial sphere packing contains

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}[i]) : b, c \in 2\mathbb{Z}[i] \right\},$$

and this subgroup maps a particular sphere to itself. This implies that the set of bends contains “primitive” values of a quaternary quadratic polynomial.

- 2 The quaternary quadratic polynomial gives you enough to work with so that you can use the circle method to give an asymptotic formula involving a singular series.
- 3 Show that the singular series (with the coprimality restriction) is bounded away from zero when m meets the congruence restrictions for the packing.

I am working on strong asymptotic local-to-global principles for certain integral Kleinian sphere packings.

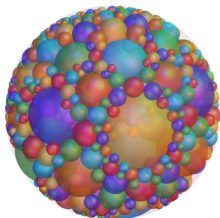


Figure: An integral orthoplicial sphere packing. Image by Kei Nakamura.

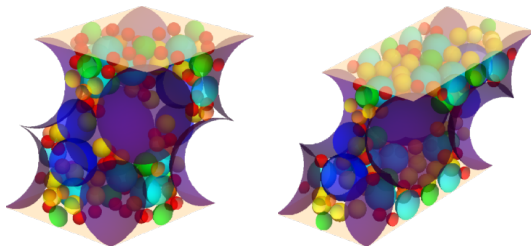


Figure: A fundamental domains of two conformally inequivalent integral Kleinian sphere packing. Images by Arseniy (Senia) Sheydvasser.

Besides the illustrations previously credited and a few orthoplicial sphere packing illustrations created by the presenter, the illustrations for this talk came from the following paper

- Kei Nakamura, “The local-to-global principle for integral bends in orthoplicial Apollonian sphere packings,” preprint, arXiv:1401.2980

Thank you for listening!

