Möbius Transformations and the Bends and Centers of Generalized Circles, Spheres, and Hyperspheres

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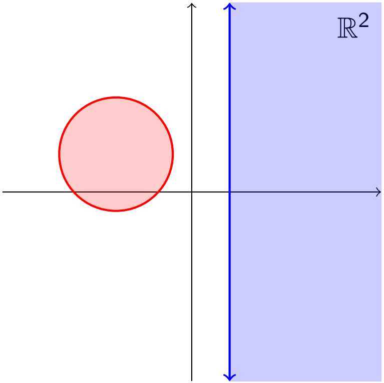
Möbius Transformations on $\mathbb{C}$

$\mathrm{SL}(2,\mathbb{C})\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$

$$z \mapsto g(z) = \frac{az+b}{cz+d}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$$



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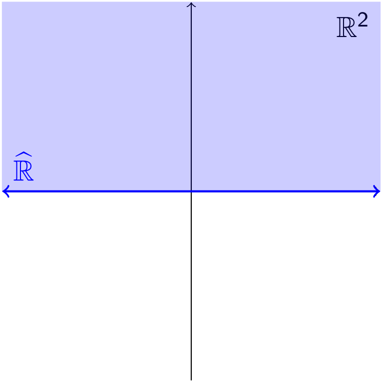

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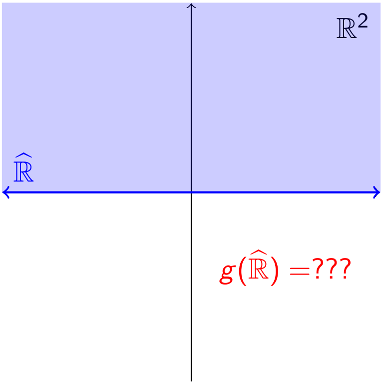

Oriented Generalized m-Spheres

Definition (Generalized m-sphere)

A *generalized m-sphere* is an m-sphere or a hyperplane in $\mathbb{R}^{m+1}$.

Examples

A generalized 1-sphere is a circle (1-sphere) or a line in $\mathbb{R}^2$.
A generalized 2-sphere is a sphere (2-sphere) or a plane in $\mathbb{R}^3$.

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Definition (Positively oriented)

An oriented m-sphere S is *positively oriented*
$\iff$ the interior of S contains the center of S.

Definition

Given an oriented generalized m-sphere S, we define the following:

- If S is not a hyperplane, then the *bend* $\beta(S)$ of S is $1/(\text{radius of } S)$, taken to be positive if S is positively oriented and negative otherwise.
  If S is a hyperplane, then its bend is $\beta(S) = 0$.

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- If S is not a hyperplane, then the *bend-center* $\xi(S) \in \mathbb{R}^{m+1}$ of S is the product of the bend $\beta(S)$ and the center of S.
  If S is a hyperplane, its bend-center is the unique unit normal vector to S pointing in the direction of the interior of S.

Inversive Coordinates

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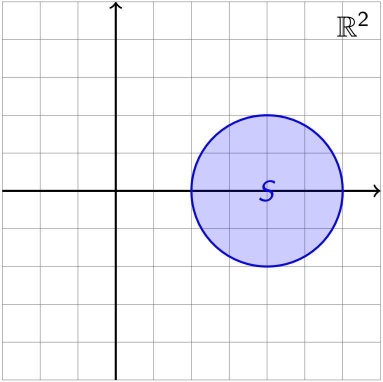
- The *inversive coordinates* of S is the ordered triple $(\beta(S), \hat{\beta}(S), \xi(S))$.

Inversive Coordinates Example 1



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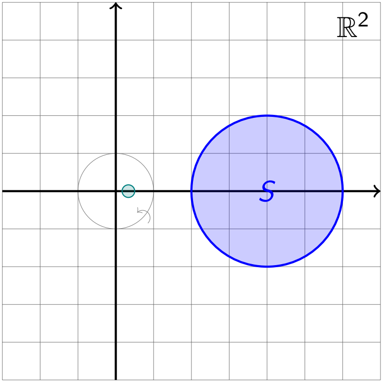

- $\beta(S) = \frac{1}{2}$

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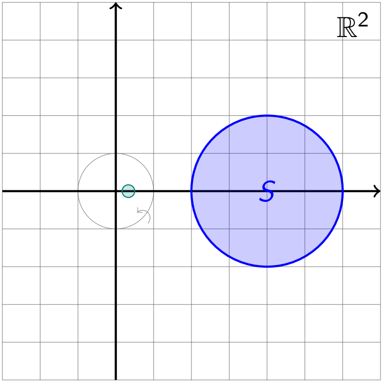

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- $\hat{\beta}(S) = 6$

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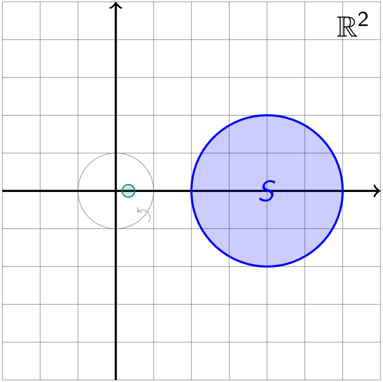
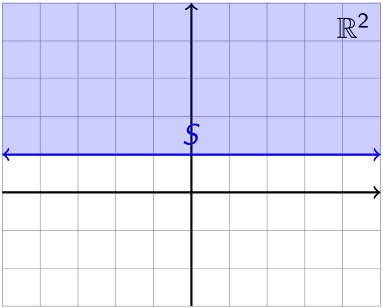


- $\beta(S) = \frac{1}{2}$
- $\hat{\beta}(S) = 6$
- $\xi(S) = \frac{1}{2}(4,0) = (2,0)$
  $\sim 2 + 0i = 2$

Inversive Coordinates Example 2



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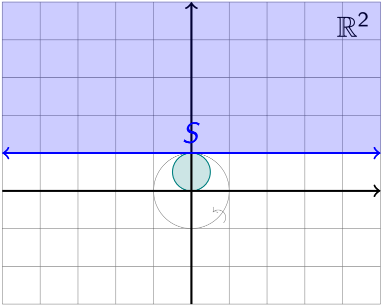

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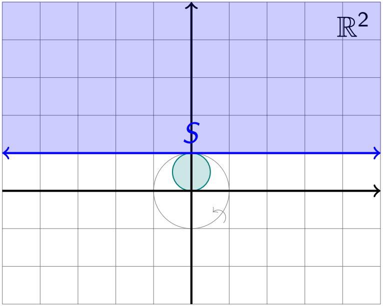

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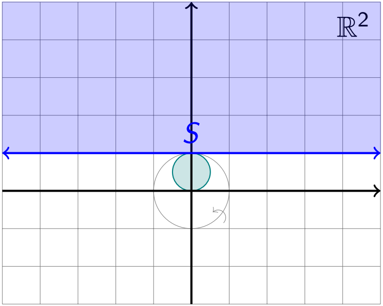


- $\beta(S) = 0$
- $\hat{\beta}(S) = 2$
- $\xi(S) = (0, 1) \sim 0 + i = i$

Inversive Coordinates Uniquely Describe an Oriented Generalized m-Sphere

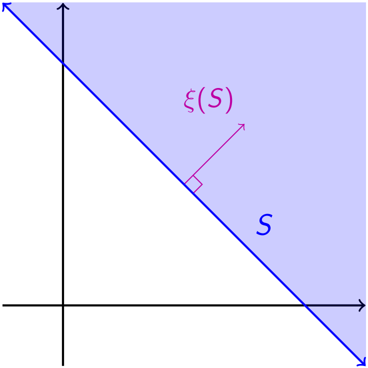
For an oriented m-sphere S,

- the radius of S is $|1/\beta(S)|$
- the center of S is $\xi(S)/\beta(S)$
- the orientation of S is indicated by the sign of $\beta(S)$

Inversive Coordinates Uniquely Describe an Oriented Generalized m-Sphere

For an oriented hyperplane S,

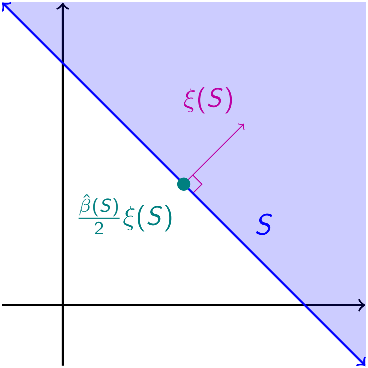
- $\xi(S)$ is the unit normal vector to S pointing in the direction of the interior of S.



Inversive Coordinates Uniquely Describe an Oriented Generalized m-Sphere

For an oriented hyperplane S,

- $\xi(S)$ is the unit normal vector to S pointing in the direction of the interior of S.
- $\frac{\hat{\beta}(S)}{2}\xi(S)$ is the closest point on S to the origin



Theorem

For an oriented generalized m-sphere S, we have

$$\beta(S)\hat{\beta}(S) - |\xi(S)|^2 = -1.$$

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Proof sketch:

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Proof sketch:

- If S is a hyperplane, then statement is true.
- If S is an m-sphere, solve for $\hat{\beta}(S)$ in terms of $\beta(S)$ and $\xi(S)$.

Theorem (Stange, 2017)

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

Then $S = g(\widehat{\mathbb{R}})$ *has the following inversive coordinates:*

- *bend* $\beta(S) = i(c\bar{d} - d\bar{c})$
- *co-bend* $\hat{\beta}(S) = i(a\bar{b} - b\bar{a})$
- *bend-center* $\xi(S) = i(a\bar{d} - b\bar{c})$

Example

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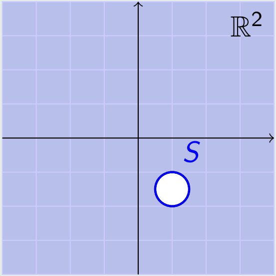
$$g = \begin{pmatrix} 2+i & 1-i \\ i & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$



Then $g(\widehat{\mathbb{R}})$ has the following inversive coordinates:

- $\beta(S) = i(i\bar{1} - 1\bar{i}) = -2$
- $\hat{\beta}(S) = i((2+i)\overline{(1-i)} - (1-i)\overline{(2+i)}) = -6$
- $\xi(S) = i((2+i)\bar{1} - (1-i)\bar{i}) = -2+3i$

- What about $g(C)$ for an arbitrary generalized circle C?

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- What about spheres in higher dimensions?

Clifford Algebras

Definition

The *Clifford algebra* C_m is the real associative algebra generated by m elements $i_1, i_2, \dots, i_m$ subject to the relations:

- $i_\ell^2 = -1 \ (1 \leq \ell \leq m)$
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Examples (Some Elements in C_m)

- $1 + i_1 + i_1 i_2 \in C_2$
- $2 + i_1 i_2 i_3 \in C_3$

Clifford Algebras

- Every $a \in C_m$ can be expressed uniquely in the form

$$a = \sum_I a_I I,$$

where the sum ranges over all products $I = i_{\nu_1} i_{\nu_2} \cdots i_{\nu_k}$, $1 \leq \nu_1 < \nu_2 < \cdots < \nu_k \leq m$, $a_I \in \mathbb{R}$, and empty product allowed

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- $C_3 \cong \mathbb{H} \oplus \mathbb{H}$

Involutions on Clifford Algebras

1. $*$: each $i_{\nu_1} i_{\nu_2} \cdots i_{\nu_k} \mapsto i_{\nu_k} \cdots i_{\nu_2} i_{\nu_1}$
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- $\bar{a} = 1 - 2i_1 - 3i_1 i_2$
- $|a|^2 = 1^2 + 2^2 + 3^2 = 14 = a\bar{a} = \bar{a}a$

Clifford Vectors

$$V_m := \{v_0 + v_1 i_1 + \cdots + v_m i_m\} \cong \mathbb{R}^{m+1}$$

$$v_0 + v_1 i_1 + \cdots + v_m i_m \leftrightarrow (v_0, v_1, \ldots, v_m)$$

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- $|v|^2 = v\bar{v} = \bar{v}v$

Clifford Group

$|v|^2 = v\bar{v} = \bar{v}v$

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- $|ab| = |a||b|$

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Definition

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in C_m$, define the *pseudo-determinant* $\Delta(g)$ as

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$$\mathrm{GL}(2, C_m) := \Big\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \Gamma_m \cup \{0\}, \\ ab^*, cd^*, c^*a, d^*b \in V_m, \Delta(g) \in \mathbb{R} \setminus \{0\} \Big\}$$

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For $g, h \in \mathrm{GL}(2, C_m)$, $\Delta(gh) = \Delta(g)\Delta(h)$.

Clifford Matrices and Möbius Transformations

$$\begin{aligned} \mathrm{GL}(2, C_m) : \widehat{V_m} &\to \widehat{V_m} \\ z &\mapsto g(z) = (az+b)(cz+d)^{-1}, \\ g &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, C_m) \end{aligned}$$

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Also act on $\widehat{V_m}$:

$$\mathrm{SL}(2, C_m) := \{g \in \mathrm{GL}(2, C_m) : \Delta(g) = 1\}$$

$$\mathrm{PSL}(2, C_m) := \mathrm{SL}(2, C_m) / \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

Clifford Matrices and Möbius Transformations

Theorem (Ahlfors, 1985)

The group $\mathrm{PSL}(2, C_m)$ *is isomorphic to the group of orientation-preserving Möbius transformations on* $\widehat{\mathbb{R}^{m+1}}$. *The group* $\mathrm{PSL}(2, C_m)$ *is generated by the matrices*

$$\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with $a \in \Gamma_m$ *and* $b \in V_m$.

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Corollary

The group $\mathrm{SL}(2, C_m)$ *is generated by the matrices*

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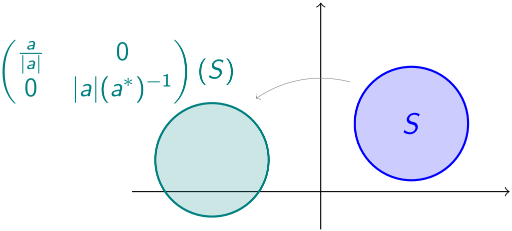
$$\begin{pmatrix} a & 0 \\ 0 & (a^*)^{-1} \end{pmatrix} : z \mapsto aza^*$$

corresponds to a rotation associated to a followed by a dilation by $|a|^2$.



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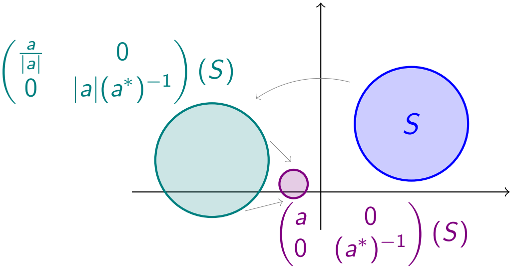
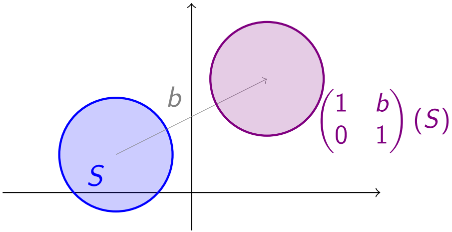


Clifford Matrices and Möbius Transformations

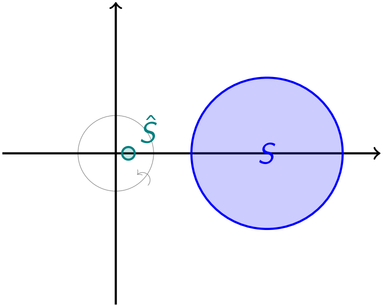
$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : z \mapsto z + b$$

corresponds to a translation by b.



$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : z \mapsto -z^{-1}.$$

corresponds to a reflection in the unit m-sphere followed by a reflection in the hyperplane $z_0 = 0$.



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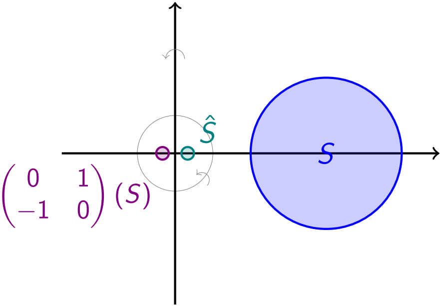

Inversive-Coordinate Matrix

Definition

Given an oriented generalized *m*-sphere S, the *inversive-coordinate matrix* of S is the 2×2 matrix

$$M_S := \begin{pmatrix} \frac{\hat{\beta}(S)}{\bar{\xi}(S)} & \xi(S) \\ & \beta(S) \end{pmatrix}.$$

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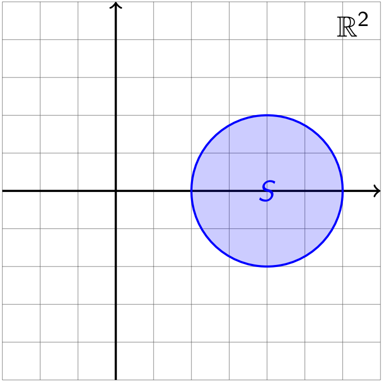
- $(\overline{M_S})^\top = M_S$
- $\Delta(M_S) = \hat{\beta}(S)(\beta(S))^* - \xi(S)(\overline{\xi(S)})^* = -1$ since

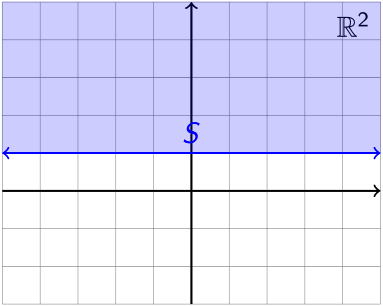
$$\beta(S)\hat{\beta}(S) - |\xi(S)|^2 = -1$$

Inversive Coordinates Example 1



- $\beta(S) = \frac{1}{2}$
- $\hat{\beta}(S) = 6$
- $\xi(S) = \frac{1}{2}(4,0) = (2,0)$
  $\sim 2 + 0i = 2$
- $M_S = \begin{pmatrix} 6 & 2 \\ 2 & \frac{1}{2} \end{pmatrix}$

Inversive Coordinates Example 2



- $\beta(S) = 0$
- $\hat{\beta}(S) = 2$
- $\xi(S) = (0, 1) \sim 0 + i = i$
- $M_S = \begin{pmatrix} 2 & i \\ -i & 0 \end{pmatrix}$

Theorem (J., 2020)

The group $\mathrm{SL}(2, C_m)$ *acts on the set of inversive-coordinate matrices by*

$$g.M := gM\overline{g}^{\top}$$

for an inversive-coordinate matrix M *and* $g \in \mathrm{SL}(2, C_m)$*. The group action of* $\mathrm{SL}(2, C_m)$ *on the set of inversive-coordinate matrices is equivalent to the group action of* $\mathrm{SL}(2, C_m)$ *on the set of oriented generalized m-spheres. That is, if* S *is an oriented generalized m-sphere and* $g \in \mathrm{SL}(2, C_m)$*, then*

$$M_{g(S)} = g.M_S.$$

Extends works that Sheydvasser did for $m = 2$ in 2019.

Proof Outline

1. Check that $g.M = gM\overline{g}^\top$ is a group action of $\mathrm{SL}(2, C_m)$ on the set of inversive-coordinate matrices.

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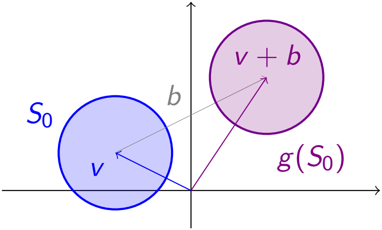
2. Verify that $M_{g(S)} = g.M_S$ for any oriented generalized m-sphere S and for any generator g of $\mathrm{SL}(2, C_m)$.

Proof Outline for Translation $z \mapsto z + b$

$$g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ for some fixed } b \in V_m \text{ and } M_{S_0} = \begin{pmatrix} \hat{\beta} & \xi \\ \overline{\xi} & \beta \end{pmatrix}.$$

$g : z \mapsto z + b$



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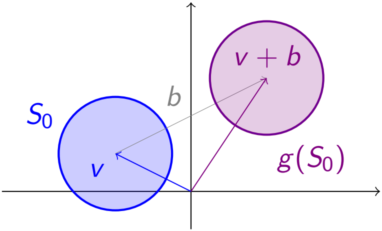

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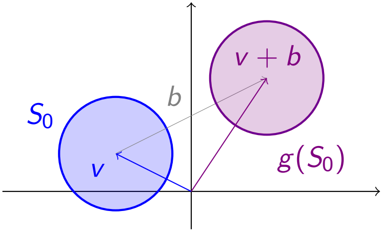

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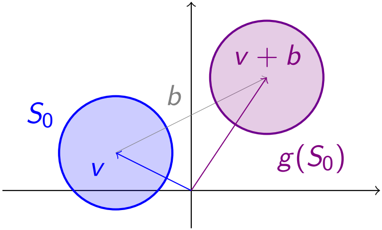

$\implies \beta \mapsto \beta$ and $\xi \mapsto \xi + \beta b$.

If $\beta \neq 0$, we can apply $\beta(S)\hat{\beta}(S) - |\xi(S)|^2 = -1$ and see that $\hat{\beta} \mapsto \hat{\beta} + b\overline{\xi} + \xi\overline{b} + \beta|b|^2$.

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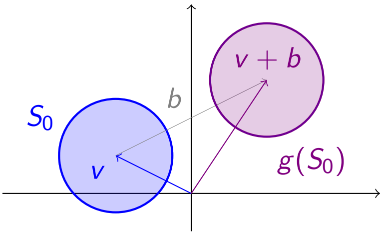

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If $\beta = 0$, we can use the fact that $\frac{\hat{\beta}(S)}{2}\xi(S)$ is the closest point on a hyperplane S to the origin and see that $\hat{\beta} \mapsto \hat{\beta} + b\overline{\xi} + \xi\overline{b} + \beta|b|^2$.

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$g = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ for some fixed $b \in V_m$.

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$\hat{\beta} \mapsto \hat{\beta} + b\overline{\xi} + \xi\overline{b} + \beta|b|^2.$

We verify that g induces the same mapping on the inversive-coordinate matrix:

$$g.M_{S_0} = gM_{S_0}\overline{g}^\top = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\beta} & \xi \\ \overline{\xi} & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \overline{b} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{\beta} + b\overline{\xi} + \xi\overline{b} + \beta|b|^2 & \xi + \beta b \\ \overline{\xi} + \beta\overline{b} & \beta \end{pmatrix}$$

Corollary (J., 2020)

Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_m),$$

and let S_0 be an oriented generalized m-sphere with an inversive-coordinate matrix

$$M_{S_0} = \begin{pmatrix} \hat{\beta} & \xi \\ \overline{\xi} & \beta \end{pmatrix}.$$

Then $g(S_0)$ has the following inversive coordinates:

- *bend* $\beta(g(S_0)) = \hat{\beta}|c|^2 + d\overline{\xi}\overline{c} + c\xi\overline{d} + \beta|d|^2$
- *co-bend* $\hat{\beta}(g(S_0)) = \hat{\beta}|a|^2 + b\overline{\xi}\overline{a} + a\xi\overline{b} + \beta|b|^2$
- *bend-center* $\xi(g(S_0)) = a\hat{\beta}\overline{c} + b\overline{\xi}\overline{c} + a\xi\overline{d} + b\beta\overline{d}$

Define $\widehat{V_{m-1}}$ be the oriented hyperplane with the inversive-coordinate matrix

$$M_{\widehat{V_{m-1}}} = \begin{pmatrix} 0 & i_m \\ -i_m & 0 \end{pmatrix}.$$

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Corollary (J., 2020)

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C_m)$.

Then $S = g(\widehat{V_{m-1}})$ *has the following inversive coordinates:*

- *bend* $\beta(S) = c i_m \bar{d} - d i_m \bar{c}$
- *co-bend* $\hat{\beta}(S) = a i_m \bar{b} - b i_m \bar{a}$
- *bend-center* $\xi(S) = a i_m \bar{d} - b i_m \bar{c}$

- $m = 1$ is Stange's result.
- $m = 2$ done by Sheydvasser.

Thank you for listening!